NUMERICAL MODELING OF INSTABILITIES IN SAND

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Masters Defense
Outline of Presentation

• Introduction
• The Constitutive Model
• Randomized porosity in FEM simulations
• Liquefaction in FEM simulations
• Conclusions
INTRODUCTION
What is an Instability?

Simplest example: **BUCKLING** of a column

The column may bow outward in any direction.

At this point, an instability has developed because the constitutive relation has bifurcated into multiple feasible solutions.

Eventually, the column will collapse due to the resulting eccentricity.
Types of Instabilities

- **GEOMETRIC** instabilities develop when bifurcation arises due to the shape of the material (e.g. buckling)

- **MATERIAL** instabilities develop when bifurcation arises due to the properties of the material itself (e.g. deformation banding)
Types of Instabilities

• **GEOMETRIC** instabilities develop when bifurcation arises due to the shape of the material (e.g. buckling)

• **MATERIAL** instabilities develop when bifurcation arises due to the properties of the material itself (e.g. deformation banding)

• Geometric instabilities are extremely rare in geomaterials; therefore, we will only consider material instabilities in this presentation
Material Instabilities

• Hill 1962 proposes that a material is unstable if a small perturbation to the body grows and stable if it does not.

• **LOCALIZED** instabilities are associated with the growth of a localized region of deformation
  
  • Example in sands: **SHEAR BANDING**

• **DIFFUSE** instabilities are NOT necessarily associated with a localized region of deformation
  
  • Example in sands: **LIQUEFACTION**
Instabilities in Sand

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SHEAR BANDING

Results in a localized region of excess deformations after Alshibli et al., 2003

LIQUEFACTION

Results in a loss of shear strength as rapid loading occurs faster than water can drain away after Kramer, 1996
Modeling Instabilities Across Scales

OUTLINE
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after Andrade et al., 2008
THE CONSTITUTIVE MODEL
The constitutive model was originally developed by Andrade and Borja, 2006 based upon the classic Nor-Sand model.

Important features of the original Nor-Sand model include:
- Critical state plasticity
- The state parameter (i.e. $\psi = \nu - \nu_c$)

New features of the Andrade and Borja, 2006 model include:
- Hyperelasticity
- A three invariant yield surface
Main Ingredients of the Constitutive Model

Three-invariant yield surface

OUTLINE
- Introduction
- THE CONSTITUTIVE MODEL
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Deviatoric Plane
Merididian Plane
Three-dimensional View
Main Ingredients of the Constitutive Model

Three-invariant yield surface - invariants

Lode Angle

Deviatoric Plane

Meridian Plane

Deviatoric Stress

Mean Normal Effective Stress

OUTLINE

- Introduction
- THE CONSTITUTIVE MODEL
  - Randomized Porosity in FEM
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Main Ingredients of the Constitutive Model

Three-invariant yield surface - shape parameters

OUTLINE
- Introduction
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Three-invariant yield surface - shape parameters

Shape Parameters

Deviatoric Plane

Meridian Plane

Figure: Three invariant yield surface on man deviatoric planes respectively.
Main Ingredients of the Constitutive Model

Three-invariant yield surface - image pressure

Image Pressure

Deviatoric Plane

Meridional Plane
Main Ingredients of the Constitutive Model

State parameter

![Graph showing state parameter ψ with CSL line and points marked at v1, vc, and v2]
Main Ingredients of the Constitutive Model

State parameter

- Image State Parameter
- Critical State Line
- Compression Gradient
- State Parameter

\[ v = \frac{\psi_1}{\lambda} \]

\[ v_c = \psi \]

\[ v_1 \]

\[ v_2 \]

\[ -\pi_i \]

\[ -p' \]
**State parameter** - significance

The state parameter influences the rate of change of the size of the yield surface because:

\[
D^* = \left( \frac{\dot{\epsilon}_p}{\dot{\epsilon}_v} \right)^* = \bar{\alpha} \psi_i
\]

Solve for the limiting image pressure

And the hardening law is:

\[
\dot{\pi}_i = h(\pi_i^* - \pi_i)\dot{\epsilon}_p
\]
**State parameter** - significance

The state parameter influences the rate of change of the size of the yield surface because:

\[
\dot{\pi}_i = h(\pi_i^* - \pi_i) \dot{\varepsilon}_s^p
\]

And the hardening law is:

\[
D^* = \begin{pmatrix} \dot{\varepsilon}_v^p & \dot{\varepsilon}_s^p \end{pmatrix} = \bar{\alpha} \psi_i
\]

Limiting Dilatancy

\[\approx 3.5 \text{ (Jefferies, 1993)}\]

Solve for the limiting image pressure

\[\pi_i^*\]
Main Ingredients of the Constitutive Model

Input parameters

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Main Ingredients of the Constitutive Model

**Input parameters** - assuming associativity and decoupling

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Easily found from laboratory experiments

Table 1: Summary of material parameters for constitutive model.
Main Ingredients of the Constitutive Model

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Can easily be determined from correlations with the friction angle.
### Main Ingredients of the Constitutive Model

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- **THE CONSTITUTIVE MODEL**
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#### Input parameters - assuming associativity and decoupling

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Must be calibrated to capture the soil the response.
Three simulations of laboratory tests were performed to evaluate the constitutive model under homogeneous conditions

1. Calibration of true triaxial tests on dense sand
   **PURPOSE:** *to evaluate the robustness of the model*

2. Calibration/prediction of TXC/PS tests on loose and dense sand
   **PURPOSE:** *to evaluate the predictiveness of the model*

3. Calibration/prediction of static liquefaction for undrained TXC
   **PURPOSE:** *to demonstrate the robustness and predictiveness of the model for undrained conditions*
Results generally show close agreement between the laboratory results and the model calibrations. **BUT**

The model has a little trouble capturing both TXC and TXE simultaneously.
True triaxial testing of loose and dense Brasted Sand

- Only the hardening parameter, h, was allowed to vary between the loose and dense tests
- The calibration under TXC fits the data very well and the predictions under PS are also reasonable
Evaluation of the Constitutive Model

OUTLINE
- Introduction
- THE CONSTITUTIVE MODEL
- Randomized Porosity in FEM
- Liquefaction modeling in FEM
- Conclusions

Undrained TXC testing of very loose Hostun Sand

Thus, the model is also able to replicate STATIC LIQUEFACTION
RANDOMIZED POROSITY IN FEM
Motivation

OUTLINE
- Introduction
- The Constitutive Model
- RANDOMIZED POROSITY IN FEM
- Liquefaction modeling in FEM
- Conclusions

X-ray computed tomography image of porosity distribution

Meso-scale porosity distribution in an FEM simulation
Overview of the FEM Testing Program

OUTLINE
- Introduction
- The Constitutive Model
- RANDOMIZED POROSITY IN FEM
- Liquefaction modeling in FEM
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- 150 drained, plane strain compression simulations were performed on 5x10 cm specimens using 20 x 40 quadrilateral isoparametric elements

- Simulations account for combinations of:
  - Anisotropy Ratios- 1, 10, 100
  - Anisotropy Orientations- 0°, 30°, 45°, 60°, 90°
Overview of the FEM Testing Program

• 150 drained, plane strain compression simulations were performed on 5x10 cm specimens using 20 x 40 quadrilateral isoparametric elements

• Simulations account for combinations of:
  • Anisotropy Ratios- 1, 10, 100
  • Anisotropy Orientations- 0°, 30°, 45°, 60°, 90°

PURPOSE: 1. to assess the sensitivity of the PS compression test to variations in void ratio heterogeneities

2. to provide a general framework coupling elastoplasticity and geostatistical tools to assess the effects of potential heterogeneities
Distribution of the Porosity Values

OUTLINE
• Introduction
• The Constitutive Model
• RANDOMIZED POROSITY IN FEM
• Liquefaction modeling in FEM
• Conclusions

Semivariogram for an isotropic simulation

Histogram of porosity distribution for a single simulation
Shear Banding with Randomized Porosity

Example: 60° anisotropy angle, anisotropy ratio=10

Initial specific volume distribution

Final deviatoric strain distribution

Shear Band
“Failure” is defined as the onset of localization (i.e. $\det A = 0$)
LIQUEFACTION MODELING IN FEM
Hill’s Instability Criterion

$$\left[ \dot{\sigma} \right] : \left[ \dot{\epsilon} \right] = 0$$
Formulation of the Liquefaction Criterion

Hill’s Instability Criterion

\[
\begin{bmatrix} \dot{\sigma} \end{bmatrix} : \begin{bmatrix} \dot{\varepsilon} \end{bmatrix} = 0
\]

+ Terzaghi’s Effective Stress Equation

\[
\sigma' = \sigma + p1
\]
Formulation of the Liquefaction Criterion

Hill’s Instability Criterion

\[
[\dot{\sigma}] : [\dot{\varepsilon}] = 0
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Terzaghi’s Effective Stress Equation

\[
\sigma' = \sigma + p1
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\[
[\dot{\sigma}] : [\dot{\varepsilon}] = [\dot{\varepsilon}] : c^{ep} : [\dot{\varepsilon}] - [\dot{p}]1 : [\dot{\varepsilon}] = 0
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Formulation of the Liquefaction Criterion

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Assumptions

- Incompressible fluids and solids
- Small strains
- Two invariant plasticity

OUTLINE

• Introduction
• The Constitutive Model
• Randomized Porosity in FEM
• LIQUEFACTION MODELING IN FEM
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**Hill’s Instability Criterion**

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[\dot{\sigma}] : [\dot{\varepsilon}] = [\dot{\varepsilon}] : \mathbf{c}^{ep} : [\dot{\varepsilon}] - [\dot{\rho}] 1 : [\dot{\varepsilon}] = 0
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**Assumptions**

- Incompressible fluids and solids
- Small strains
- Two invariant plasticity
LIQUEFACTION CRITERION

\[ H - H_{crit} = 0 \]

where

\[ H_{crit} = -K \frac{\partial F}{\partial p'} \frac{\partial Q}{\partial p'} \]

\[ H = -\frac{1}{\lambda} \frac{\partial F}{\partial \pi_i} \dot{\pi}_i \]

Example of the evolution of \( H \) and \( H_{crit} \) at a Gauss point
Formulation of the Liquefaction Criterion

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- Introduction
- The Constitutive Model
- Randomized Porosity in FEM
- LIQUEFACTION MODELING IN FEM
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LIQUEFACTION CRITERION

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\[ H = -\lambda \frac{1}{\partial \pi_i} \frac{\partial F}{\partial \pi_i} \]

Example of the evolution of \( H \) and \( H_{crit} \) at a Gauss point

Onset of Liquefaction

Critical Hardening Modulus
For the simulations that will follow, one of the assumptions used in the above formulation is not valid!

The constitutive model has **THREE INVARIANTS** (not two)
Important Caveat

For the simulations that will follow, one of the assumptions used in the above formulation is not valid!

The constitutive model has **THREE INVARIANTS** (not two)

However, comparisons with a more general criterion developed by Borja, 2006 show that both methods are nearly identical:

\[
\text{det } L = 0 \quad \text{where} \quad L = \begin{bmatrix}
    c_{11}^{ep} & c_{12}^{ep} & c_{13}^{ep} & -1 \\
    c_{21}^{ep} & c_{22}^{ep} & c_{23}^{ep} & -1 \\
    c_{31}^{ep} & c_{32}^{ep} & c_{33}^{ep} & -1 \\
    -1 & -1 & -1 & 0
\end{bmatrix}
\]
Evolution of the Liquefaction Criterion

Example: Undrained plane strain compression test

\[ H-H_{\text{crit}} \]
Evolution of the Liquefaction Criterion

Example: Undrained plane strain compression test

\[ H - H_{\text{crit}} \]
Outlines:

- Introduction
- The Constitutive Model
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- Liquefaction Modeling in FEM
- Conclusions

- Liquefaction Criterion
- Deviatoric Strains
- Pore Pressures
OUTLINE
• Introduction
• The Constitutive Model
• Randomized Porosity in FEM
• LIQUEFACTION MODELING IN FEM
• Conclusions

Undrained Plane Strain Compression Simulation

- Liquefaction Criterion

- DEVIATORIC STRAINS

- Pore Pressures
Undrained Plane Strain Compression Simulation

- Liquefaction Criterion
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- PORE PRESSURES
OUTLINE
• Introduction
• The Constitutive Model
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Increasing Pore Pressure Gradient Simulation

(a) $\Delta p = 37$ kPa
(b) $\Delta p = 55$ kPa
(c) $\Delta p = 95$ kPa
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• Introduction
• The Constitutive Model
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Submarine Slope Simulation

Dimensions and loading conditions

Simulation results at the onset of liquefaction
Loading Rate Effects

Effect of loading rate on drained PS compression tests

The loading rate can be normalized in terms of displacement rate and hydraulic conductivity:

\[ Z = \frac{\dot{u}y}{k} \]

But, Z only applies when the loading is unidirectional.
Effect of mesh size on undrained PS compression tests

Mesh Size Effects

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Effect of mesh size on undrained PS compression tests

Mesh Size Effects
CONCLUSIONS
Main Conclusions

• The Andrade and Borja, 2006 constitutive model was shown to make reasonably accurate and robust predictions of sand behavior under monotonic loading conditions.

• Minute variations in porosity at the meso-scale can account for a relatively wide range of compressive strengths due to its heavy influence on shear band formation.

• The onset and location of liquefaction can be tracked in finite element analyses using recently developed criteria.

• A unified constitutive framework can distinguish between different modes of instabilities.
ACKNOWLEDGMENTS...