Criteria for Static Equilibrium in Discrete Element Methods

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State-of-the-Art Static DEM
Explicit Scheme & Numerical Calibration

Governing equation: \[ f^{\text{int}} = Ku = f^{\text{ext}} \]

Solution scheme (explicit):
\[ m\ddot{u} + D\dot{u} + Ku = f^{\text{ext}} \]

- **Numerical calibration:**
  - Numerical parameters: fictitious mass \( m \), damping \( D \), time step \( \Delta t \), loading rate, etc.
  - Must be calibrated to achieve quasi-static state

- **Equilibrium criterion:**
  - Governs numerical calibration

- **Problem:**
  - \( f^{\text{int}} \) and \( f^{\text{ext}} \) depend on assembly size
**Static Equilibrium of Particulate Systems**

**Homogenized stress**
(Christoffersen et al. 1981):

\[
\bar{\sigma} = \frac{1}{V} \sum_{\alpha=1}^{N_c} f^\alpha \otimes d^\alpha
\]

**From static force balance**

**external stress:**

\[
\bar{e} = \frac{1}{V} \sum_{p=1}^{N_p} b^p \otimes x^p + \frac{1}{V} \sum_{\beta=1}^{N_t} t^\beta \otimes x^\beta
\]

\[
\bar{\sigma} = \bar{e}
\]

**From static moment balance**

(Bardet & Vardoulakis, 2001)

\[
\sum_{\alpha=1}^{N_c} f^\alpha \otimes d^\alpha = \sum_{\alpha=1}^{N_c} d^\alpha \otimes f^\alpha
\]

\[
\bar{\sigma} = \bar{\sigma}^T
\]
Numerical Calibration for Static DEM

- (1) Computation setup

- Vertical: strain-controlled
- Horizontal: $\sigma_3=50$ kPa
- Packing: $\alpha=30^\circ$, i.e., dense packing
- Intergranular friction: $\phi_\mu=30^\circ$
- No end friction, no cohesion

\[
\frac{\sigma_1}{\sigma_3}_{\text{max}} = \cot(\alpha - \phi_\mu) \cot \alpha
\]

(after Rowe, 1962)

$\sigma_{1\text{max}} = 352$ kPa
Numerical Calibration for Static DEM

- (2) Benchmark case

<table>
<thead>
<tr>
<th>Parameters:</th>
<th>$\Delta t$</th>
<th>$D_t$</th>
<th>$D_r$</th>
<th>$N_{step}$ (0.18%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark Values:</td>
<td>$0.4\sqrt{2m/(5k_r)}$</td>
<td>$2\sqrt{m/k_n}$</td>
<td>$2r^2\sqrt{2mk_r/5}$</td>
<td>$3e5$</td>
</tr>
</tbody>
</table>

\[
\bar{\sigma} = \bar{b}
\]

\[
\bar{\sigma} = \bar{\sigma}^T
\]
Numerical Calibration for Static DEM

- (3) effect of time step $\Delta t$

$\Delta t = 1.5 \Delta t^{bm}$

$\Delta t = 0.1 \Delta t^{bm}$
Numerical Calibration for Static DEM

- (4) translation damping $D_t$

\[ D_t = 0.7 D_t^{bm} \]

\[ D_t = 4.3 D_t^{bm} \]
Numerical Calibration for Static DEM

- (5) rotation damping $D_r$

$$D_r = D_{r}^{bm}$$

$$D_r = 0$$

$$D_r = 2D_{r}^{bm}$$
Numerical Calibration for Static DEM

- (6) Effect of assembly size

![Graph showing the effect of assembly size on vertical stress and computation steps.](image)
Concluding Remarks

- Explicit DEM scheme needs to be calibrated for static problems

- Homogenized stress useful for equilibrium evaluation
  - Static force balance: $\overline{\sigma} = \overline{e}$
  - Static moment balance: $\overline{\sigma} = \overline{\sigma}^T$

- Numerical Calibration
  - Time step ($\Delta t$) has lower bound
  - Dampings ($D_t$ & $D_r$) have upper bounds
  - $D_r$ is important
  - Numerical calibration can be performed on a smaller assembly, then upscaled to prototype