Finite element method: the basics

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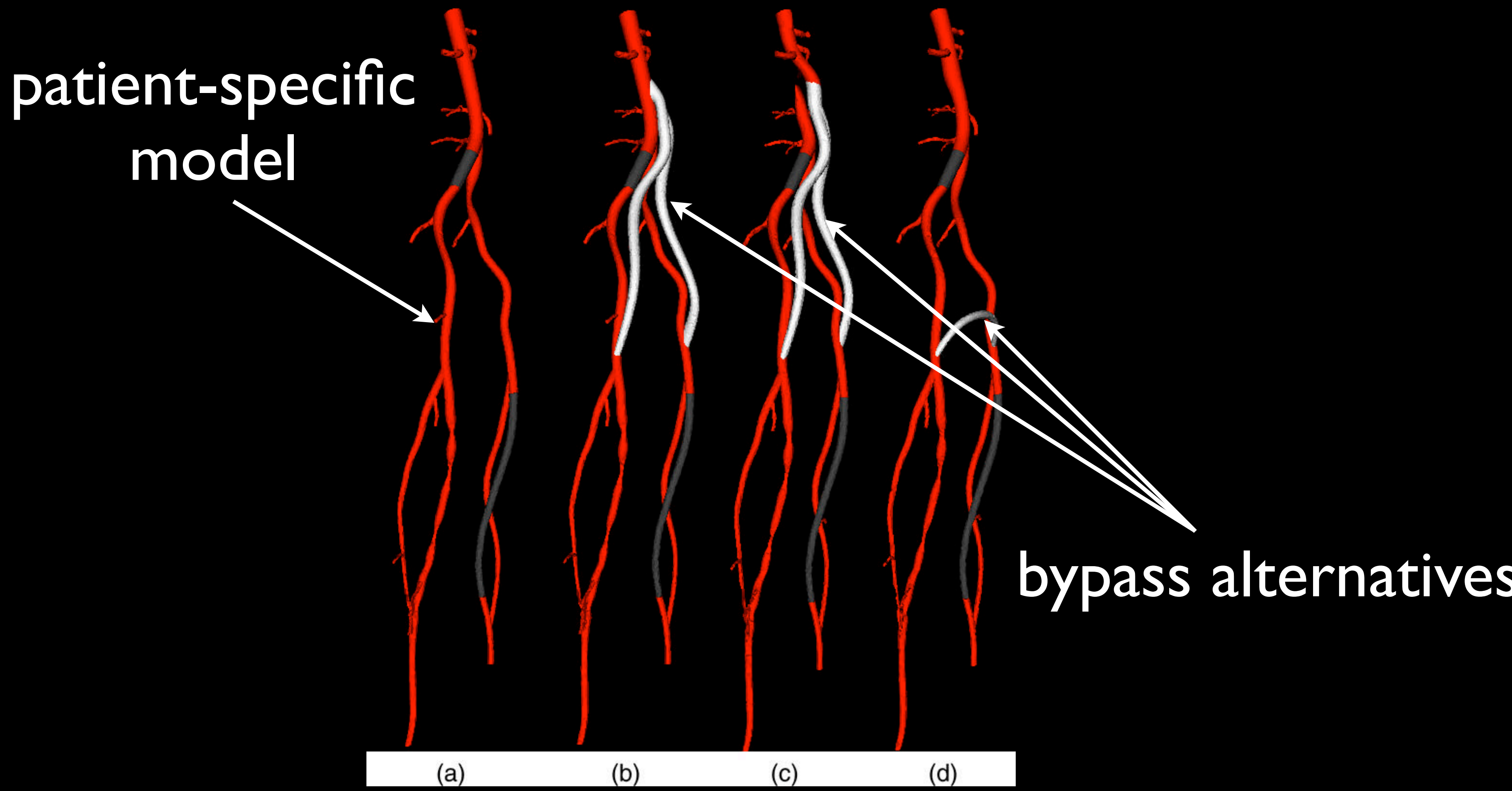
PLAXIS

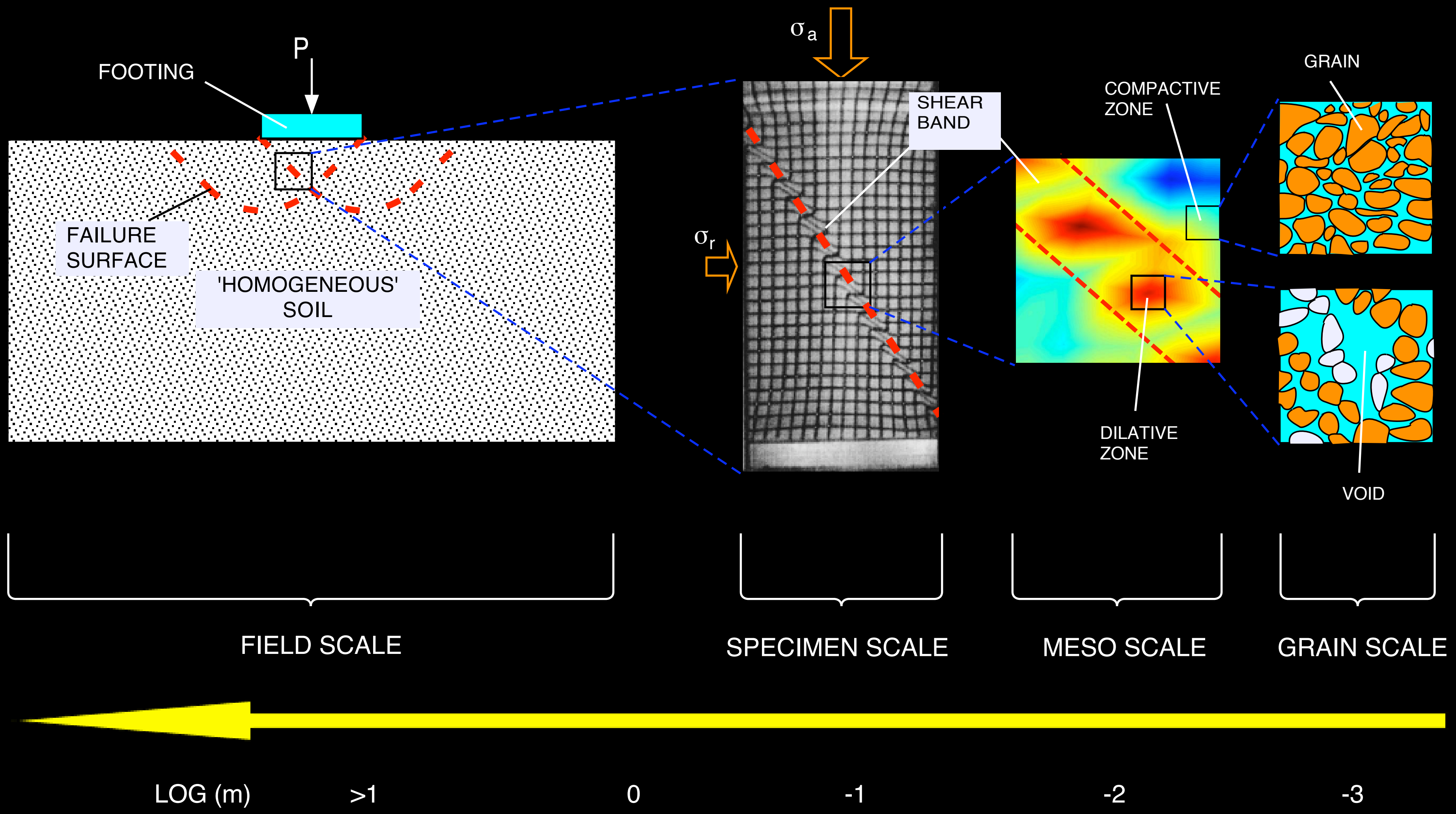
Outline

- Applications of FEM
- Fundamentals
- Examples

Applications of FEM

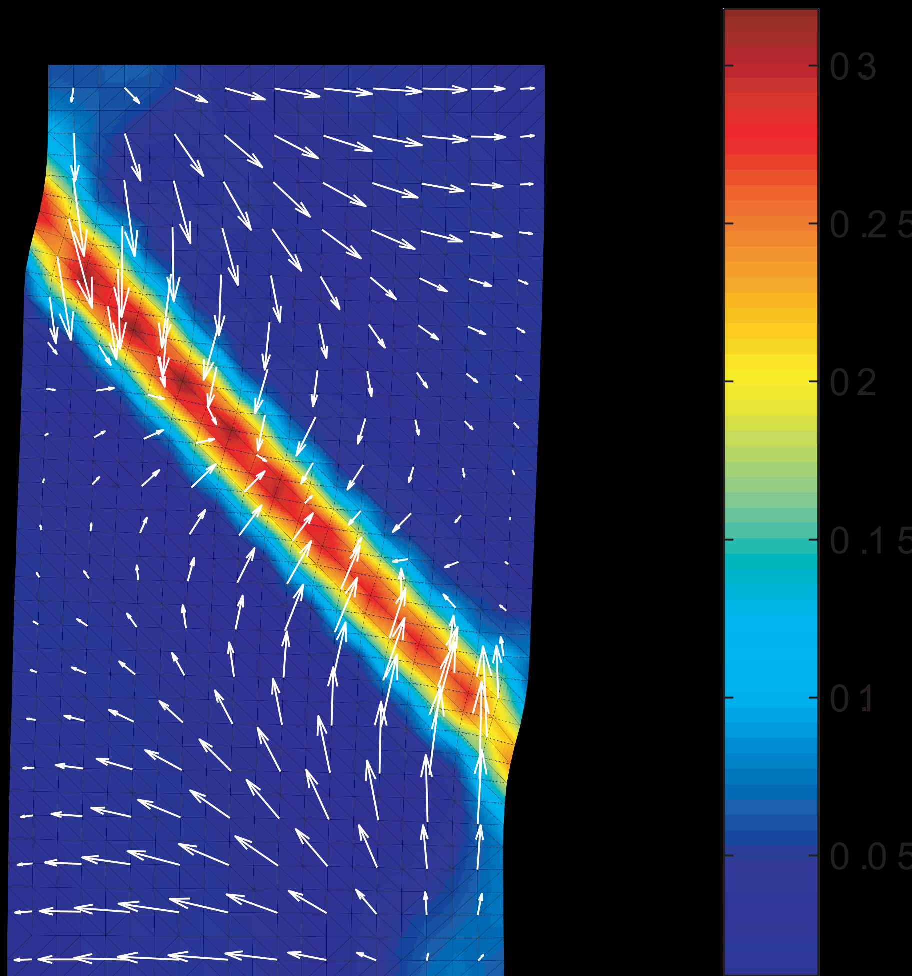


Biomechanics

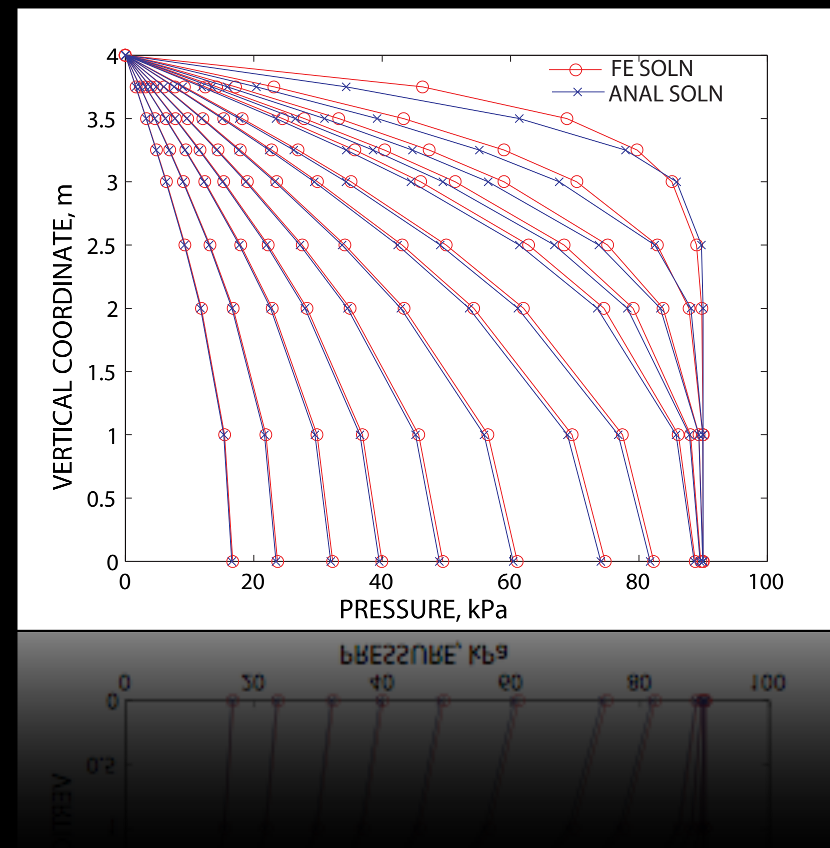


Geomechanics

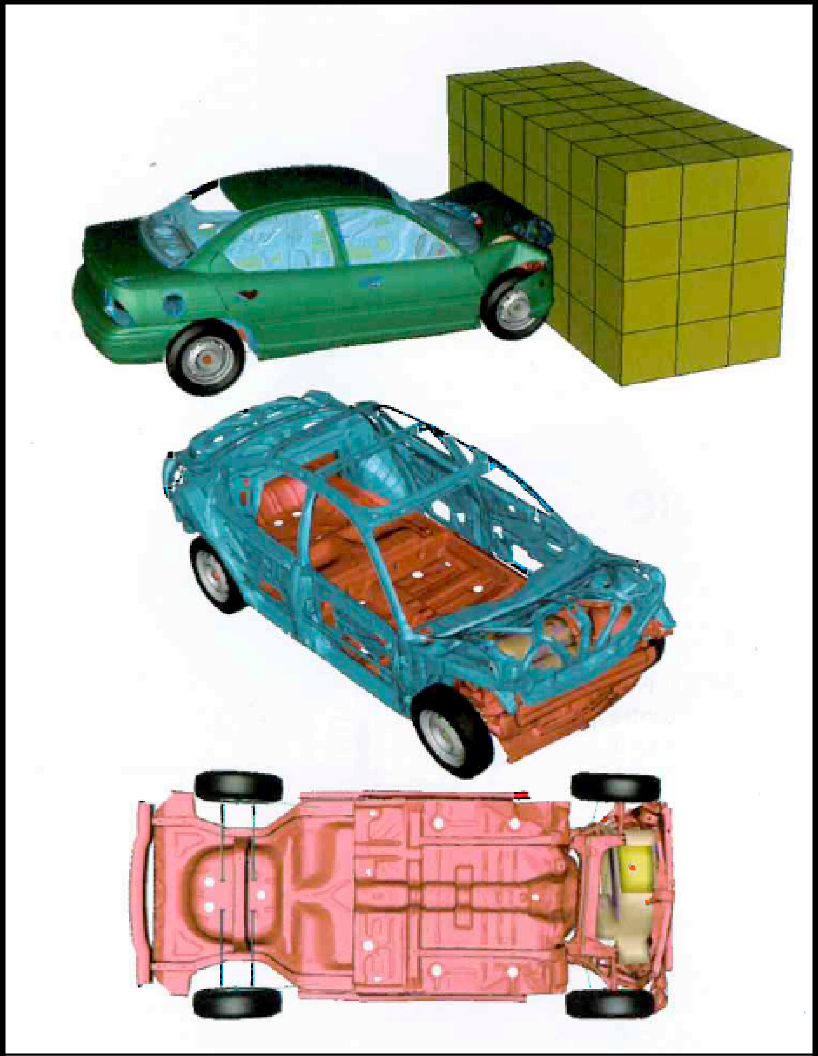
shear strain



consolidation



Solid-fluid interactions

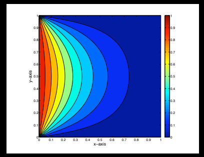


Simulation engineering

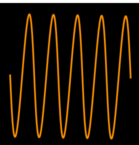
Fundamentals of FEM

FEM

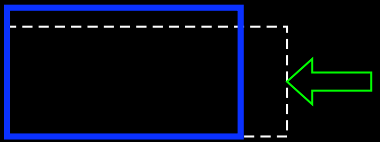
- Designed to *approximately* solve PDE's
- PDE's model physical phenomena
- Three types of PDE's:
    - Parabolic: fluid flow



    - Hyperbolic: wave eqn



    - Elliptic: elastostatics



FEM recipe

Strong from

↕

Weak form

↓

Galerkin form

↓

Matrix form

Elastostatics: strong

$$
\left.\begin{array}{lrcl}
\text{PDE} & u_{,xx} + f & = & 0 \\
\text{B.C.'s} & \left[\begin{array}{r} u(1) \\ u_{,x}(0) \end{array}\right. & \begin{array}{c} = \\ = \end{array} & \begin{array}{c} g \\ h \end{array}
\end{array}\right\} \text{strong form}
$$

derived from continuum mechanics

strong form = PDE + B.C.'s

usually, there is no exact sln for strong form

Elastostatics: weak

- Use principle of virtual work
- Introduce virtual displacement $w; \quad w(1) = 0$
- Use strong form

$$\int_0^1 w(u_{,xx} + f)\,dx = 0$$

$$\boxed{\int_0^1 w_{,x}\, u_{,x}\; dx = \int_0^1 wf\,dx + w(0)h}$$

Elastostatics: Galerkin

like g

Construct approximate solution $$u^h = v^h + g^h$$

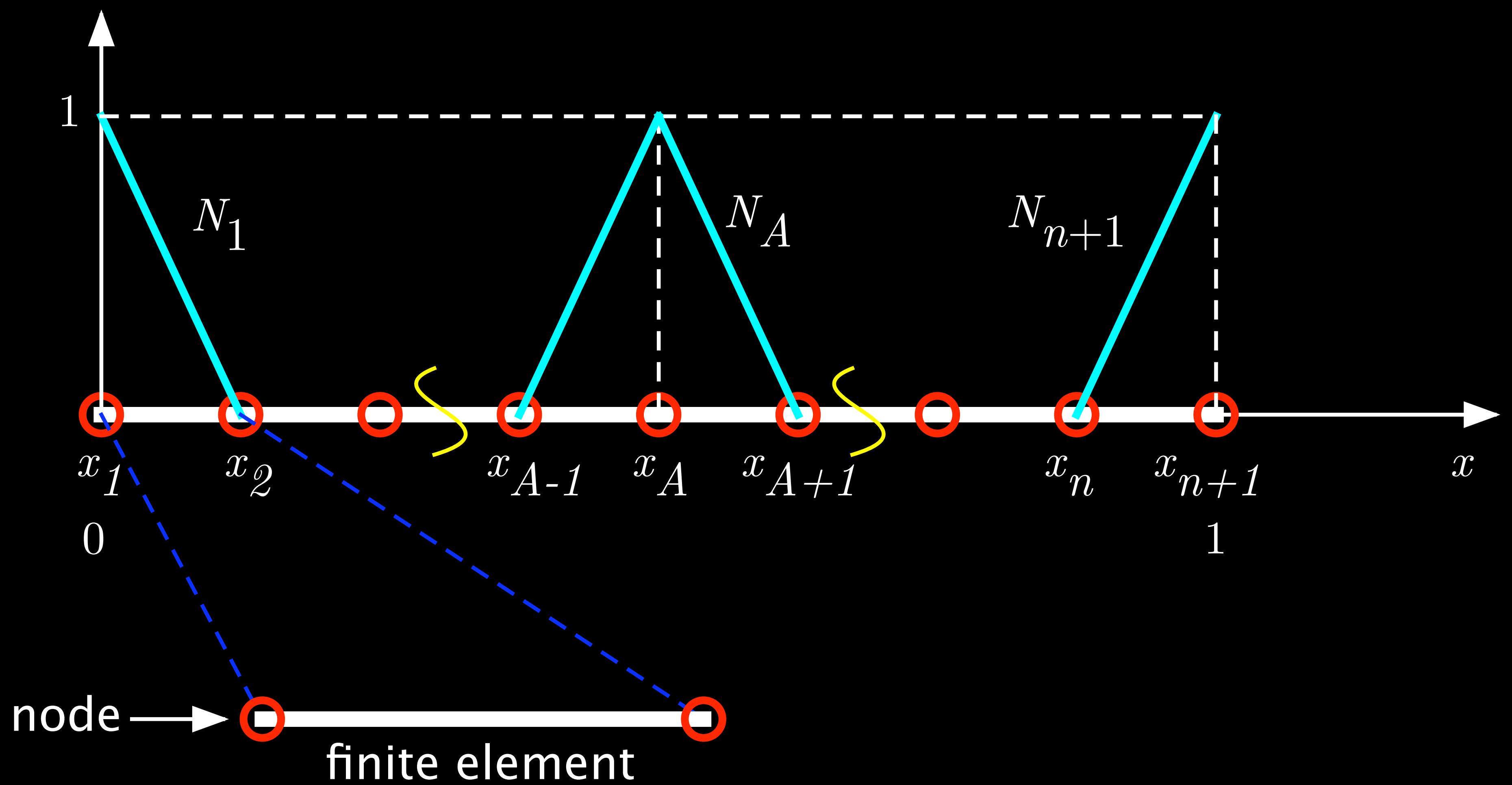
like w

Construct functions based on shape functions

$$w^h = \sum_{A=1}^{n} N_A c_A \qquad\qquad v^h = \sum_{A=1}^{n} N_A d_A$$

$$g^h = g N_{n+1}$$

Piecewise linear FE



Elastostatics: matrix

Use weak form, plug-in Galerkin approximation

$$\sum_{B=1}^{n} K_{AB} d_B = F_A \qquad K_{AB} = \int_0^1 N_{A,x} N_{b,x}\, dx$$

$$F_A = \int_0^1 N_A f\, dx + N_A(0)h - \int_0^1 N_{A,x} N_{n+1,x}\, dx g$$

it all boils down to...

$$\boxed{\boldsymbol{K} \cdot \boldsymbol{d} = \boldsymbol{F}}$$

stiffness matrix

force vector

Properties of $K \cdot \mathbf{d} = F$

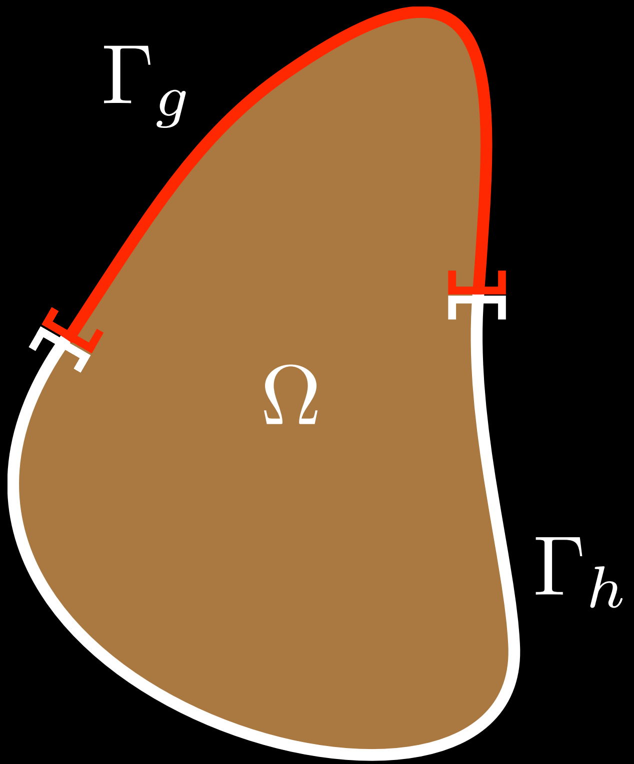
- Stiffness matrix is
  - symmetric
  - banded
  - positive-definite

may not always apply

- Displacement vector = unknowns only
- Can use any linear algebra solver to find solution

Multi-D deformation

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} &= \mathbf{0} \quad \text{in } \Omega &\longleftarrow \text{equilibrium} \\
\boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \Gamma_g &\longleftarrow \text{e.g., clamp} \\
\boldsymbol{\sigma} \cdot \boldsymbol{n} &= \boldsymbol{h} \quad \text{on } \Gamma_h &\longleftarrow \text{e.g., confinement}
\end{aligned}
$$

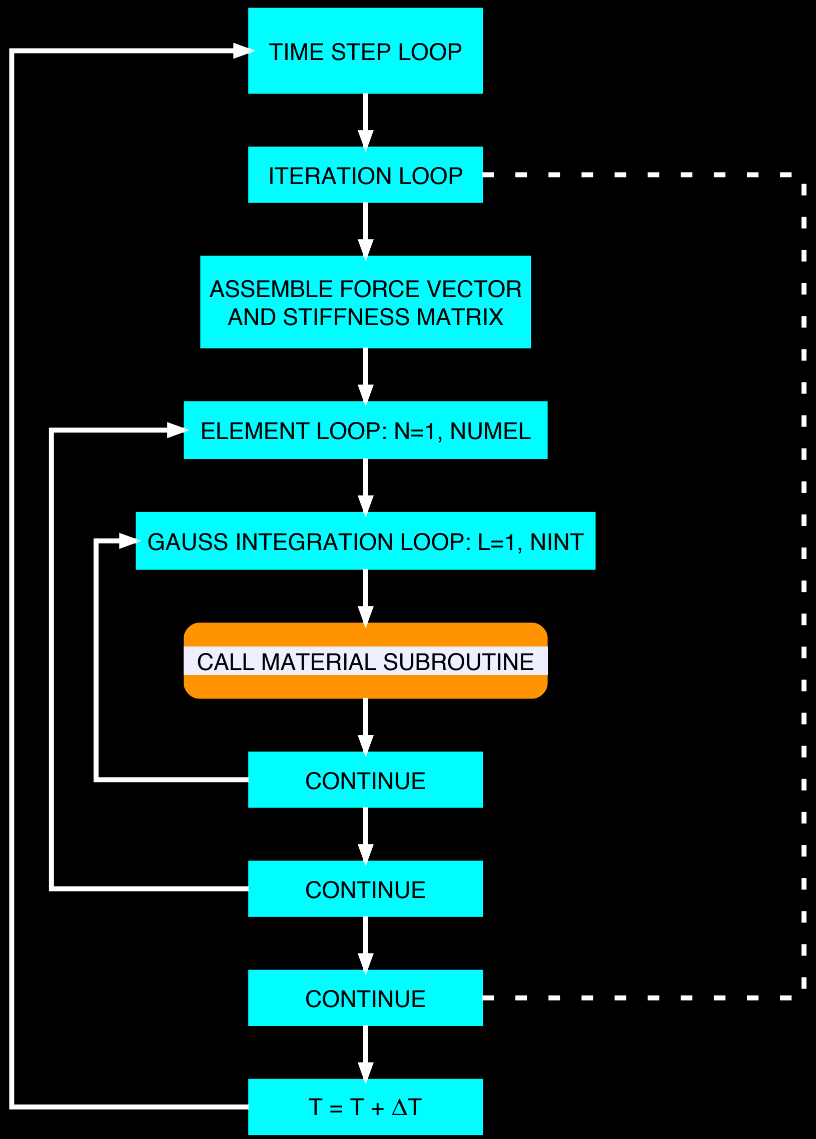


Constitutive relation

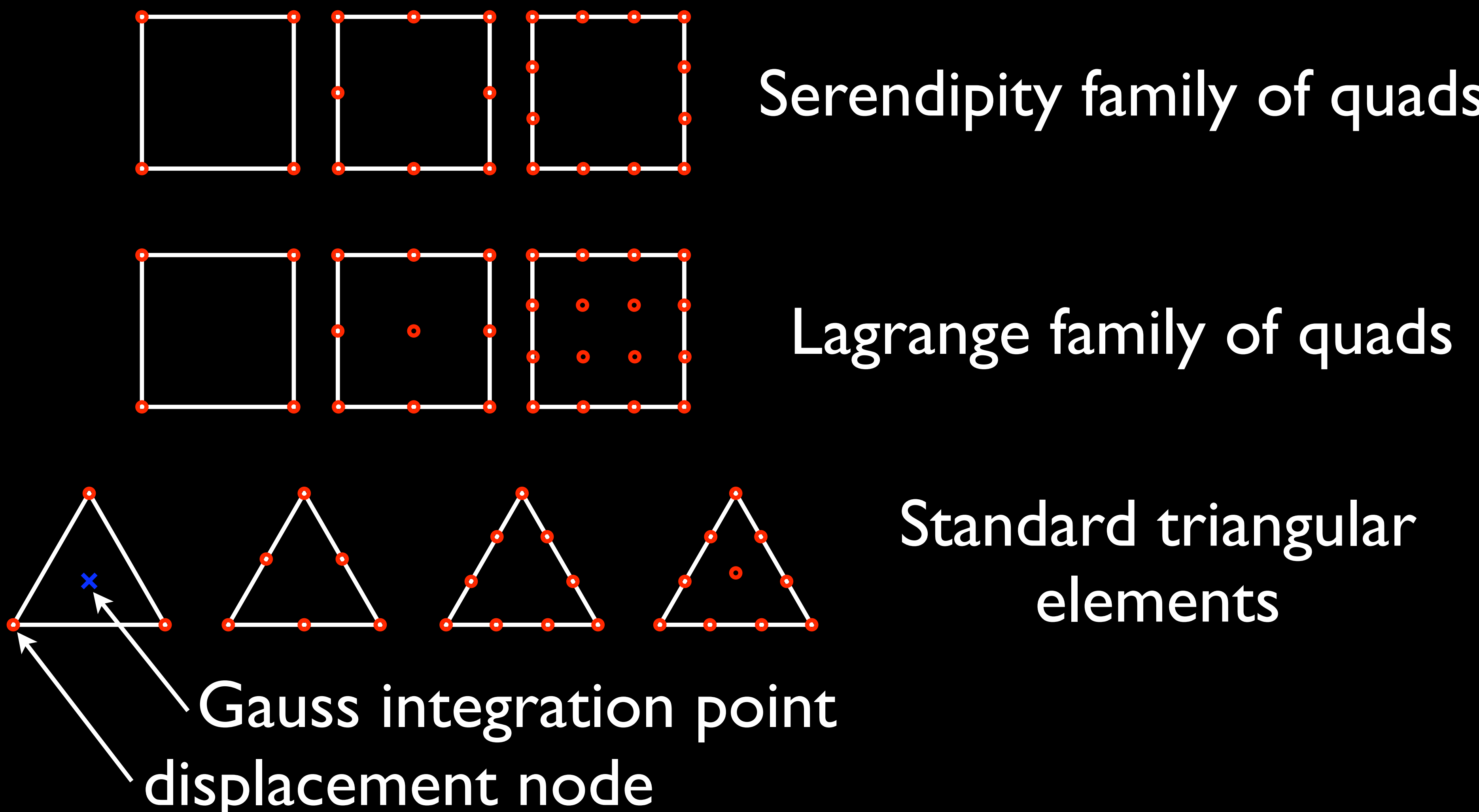
given $\boldsymbol{u}$ ⇨ get $\boldsymbol{\sigma}$

e.g., elasticity, plasticity

FEM program

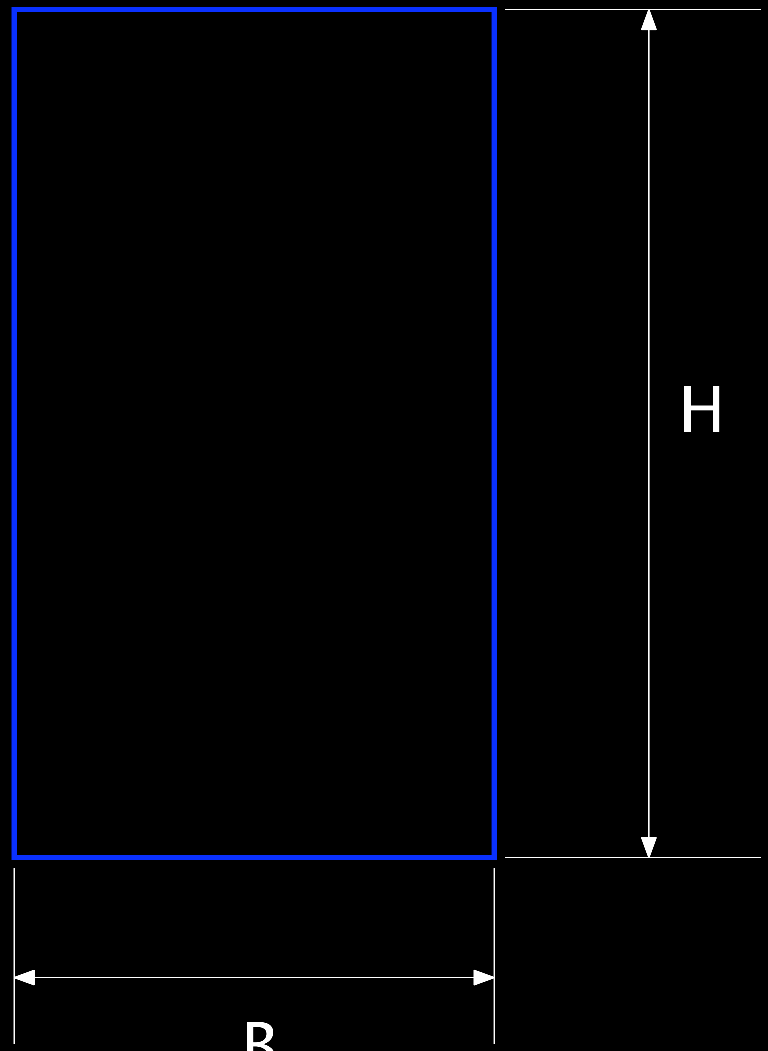

TIME STEP LOOP
ITERATION LOOP
ASSEMBLE FORCE VECTOR AND STIFFNESS MATRIX
ELEMENT LOOP: N=1, NUMEL
GAUSS INTEGRATION LOOP: L=1, NINT
CALL MATERIAL SUBROUTINE
CONTINUE
CONTINUE
CONTINUE
T = T + ΔT

Element technology: 2D



Modeling ingredients

1. Set geometry
2. Discretize domain
3. Set matl parameters
4. Set B.C.'s
5. Solve



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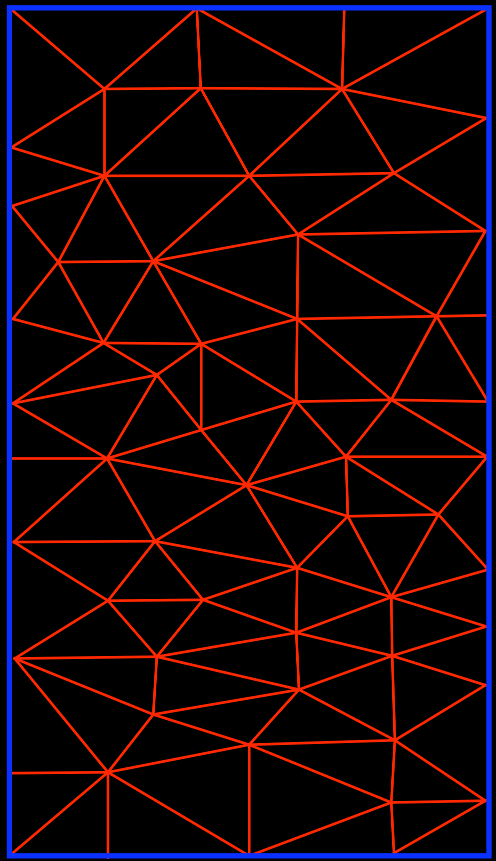

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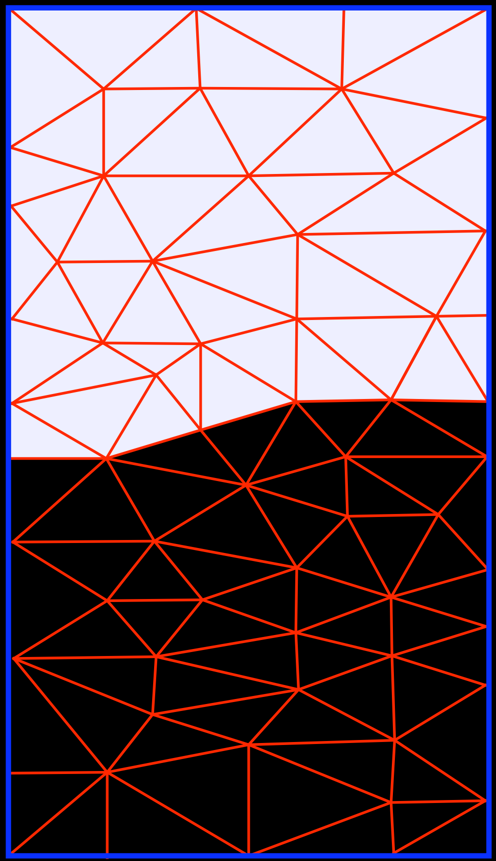

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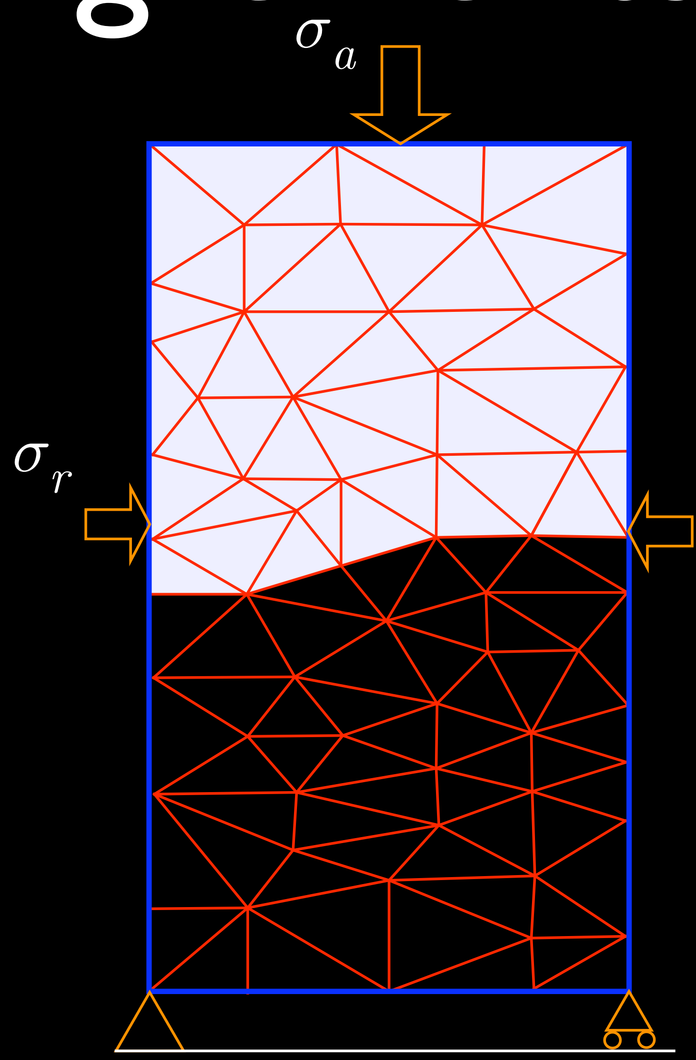

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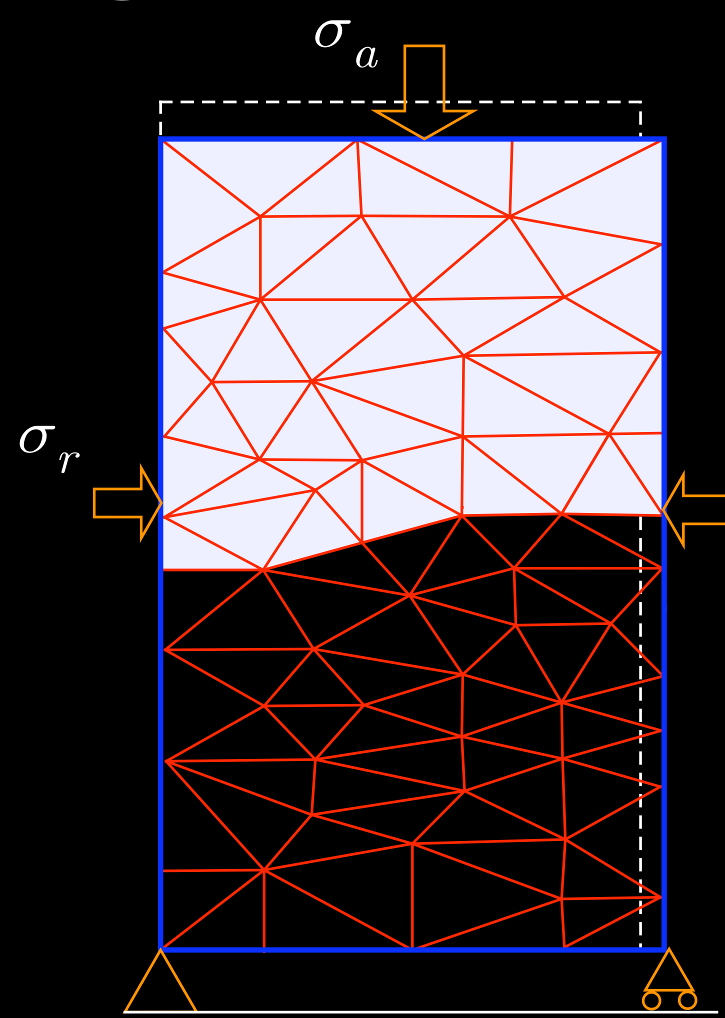
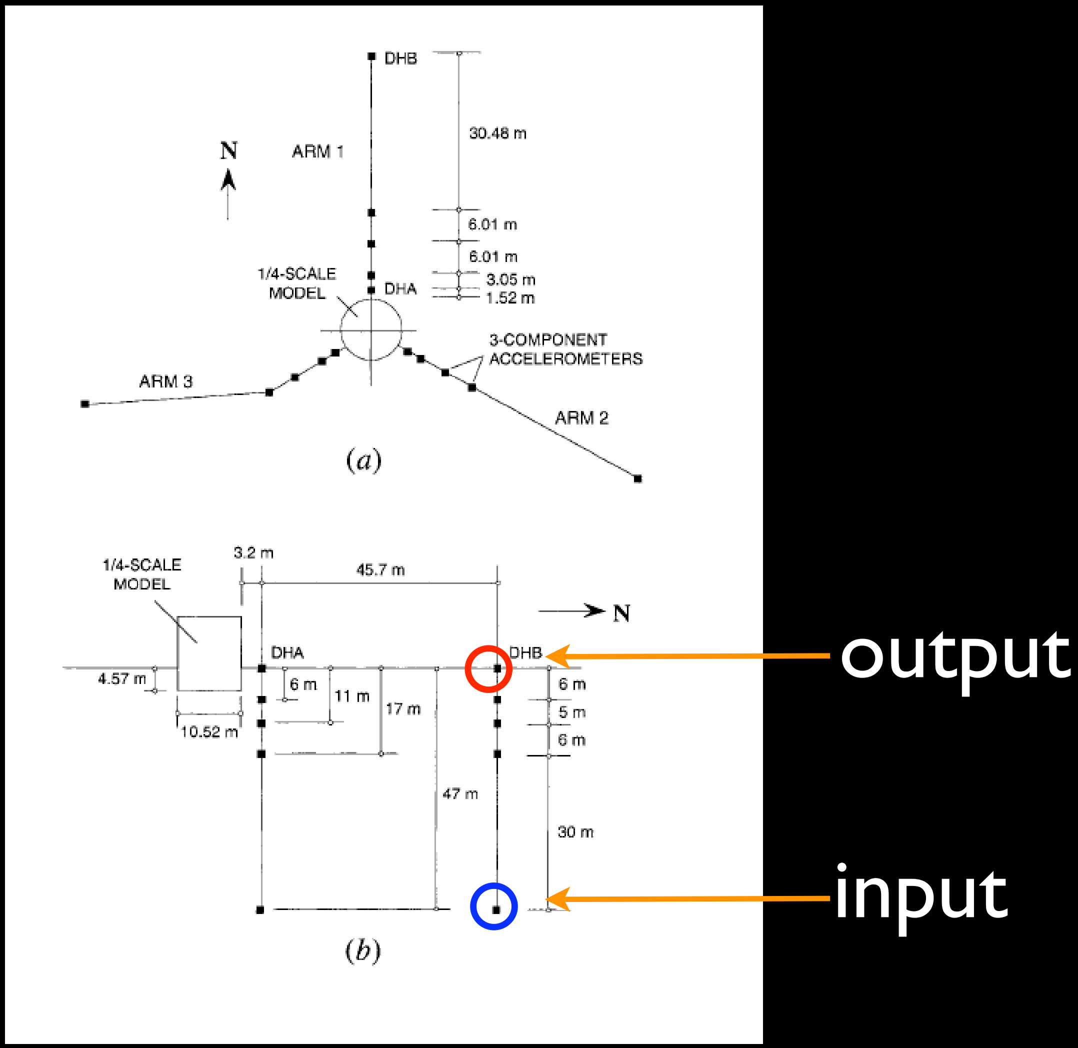
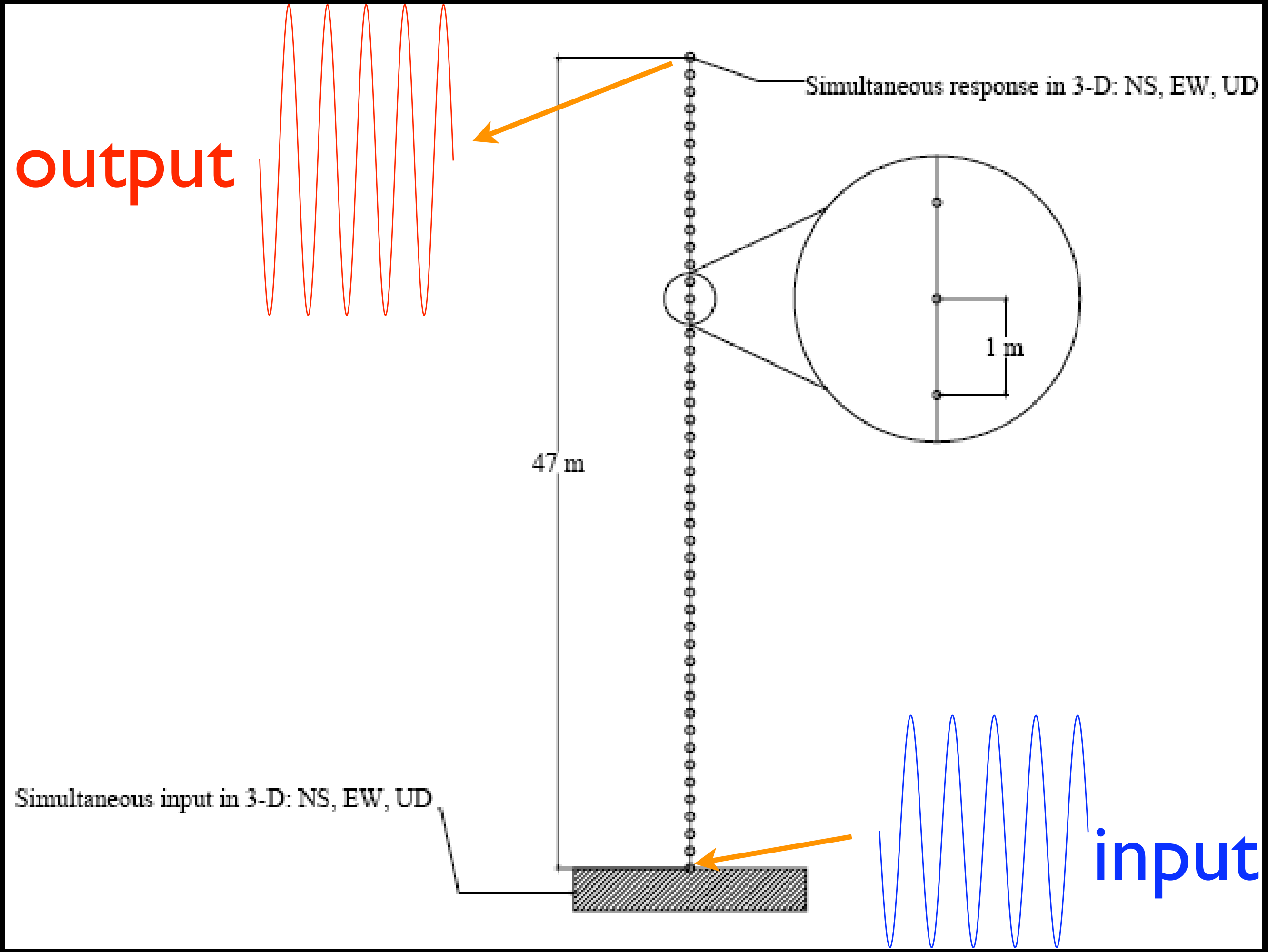


Examples

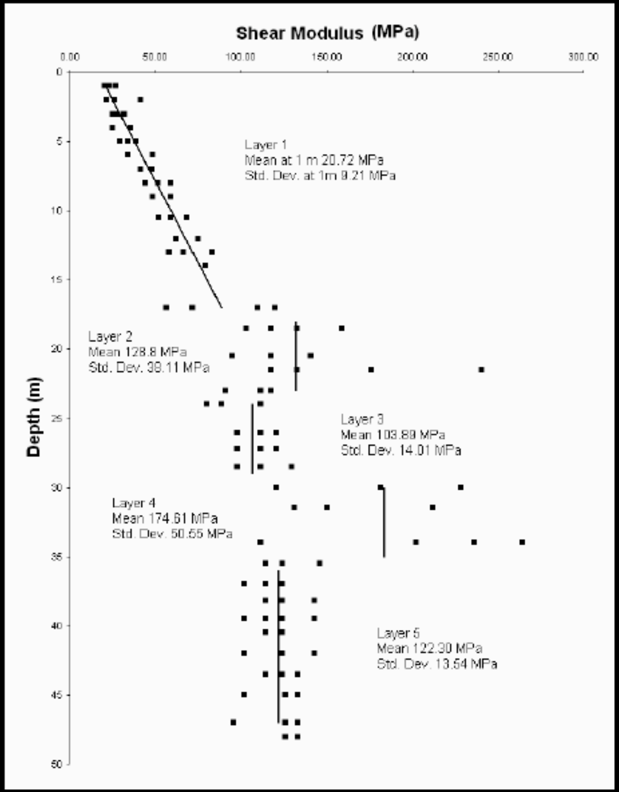
Hyperbolic: LSST array



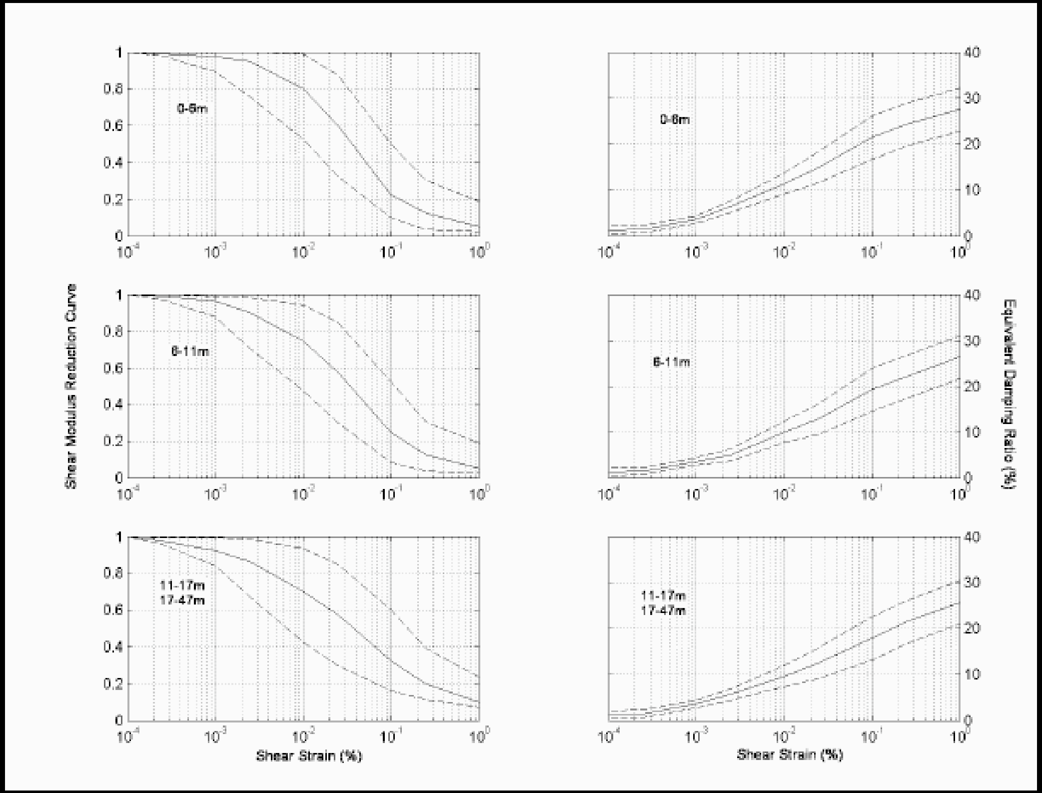
Geometry and B.C.s



Material parameters

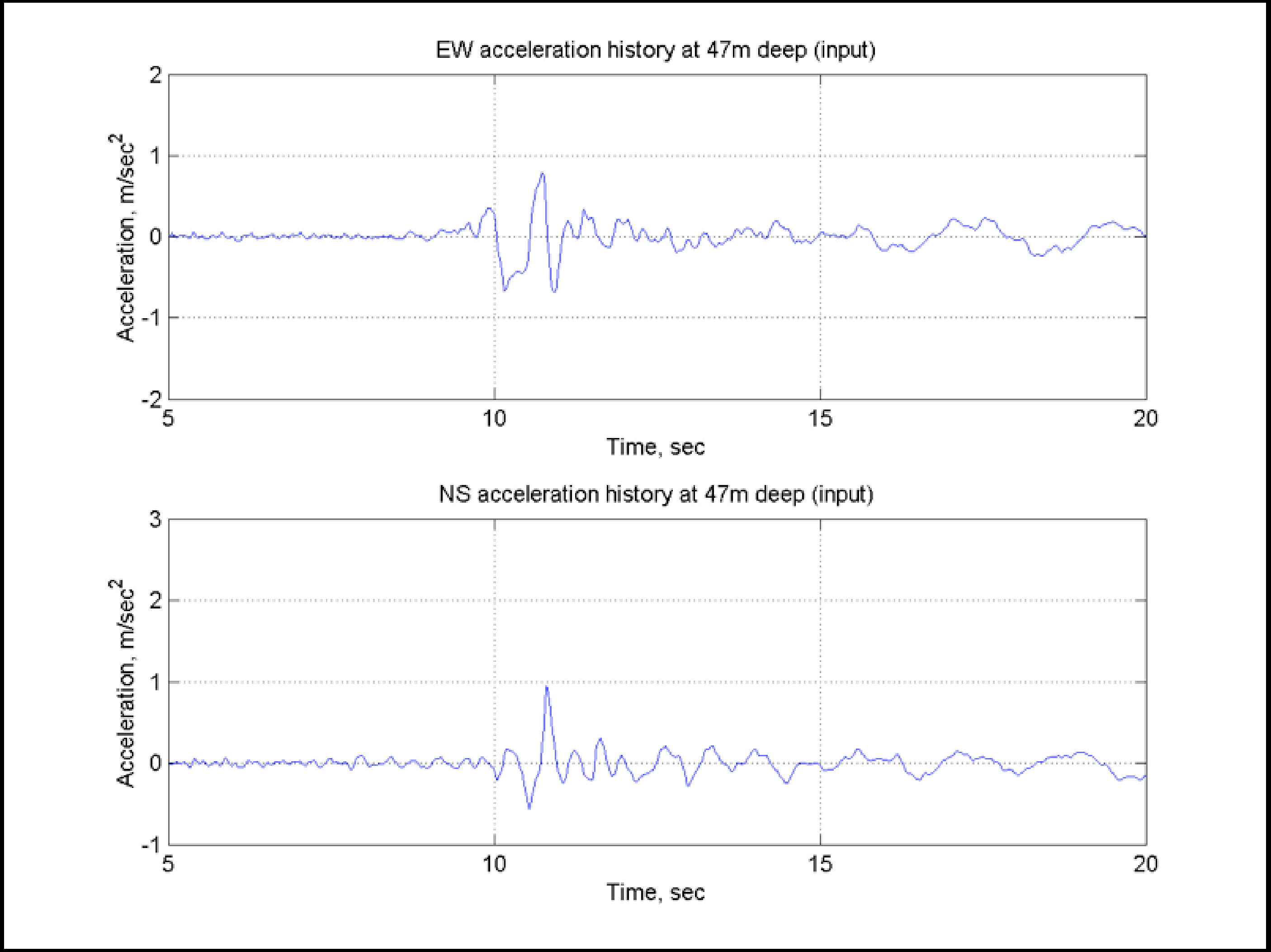


shear modulus

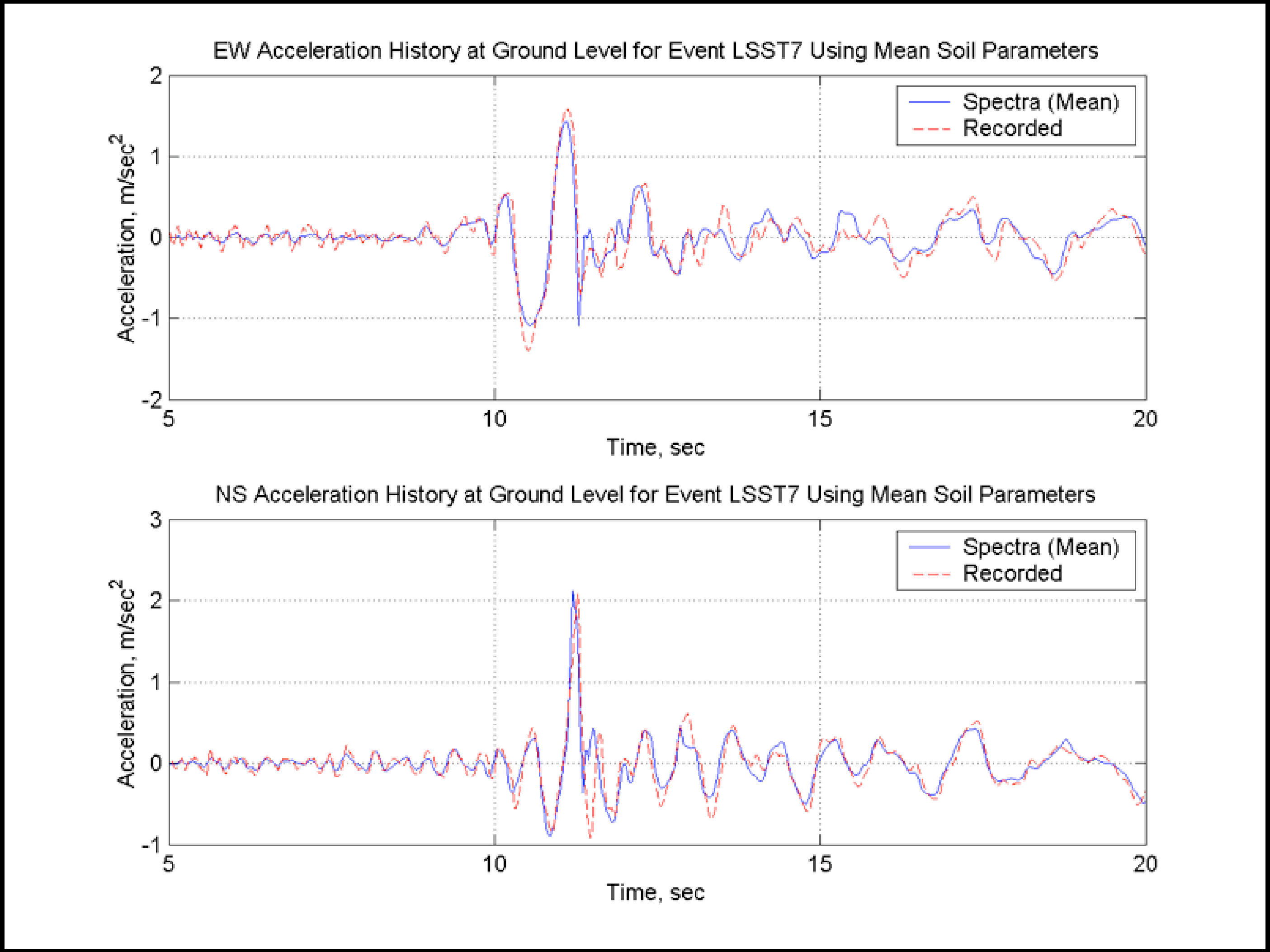


degradation & damping

Input base acceleration



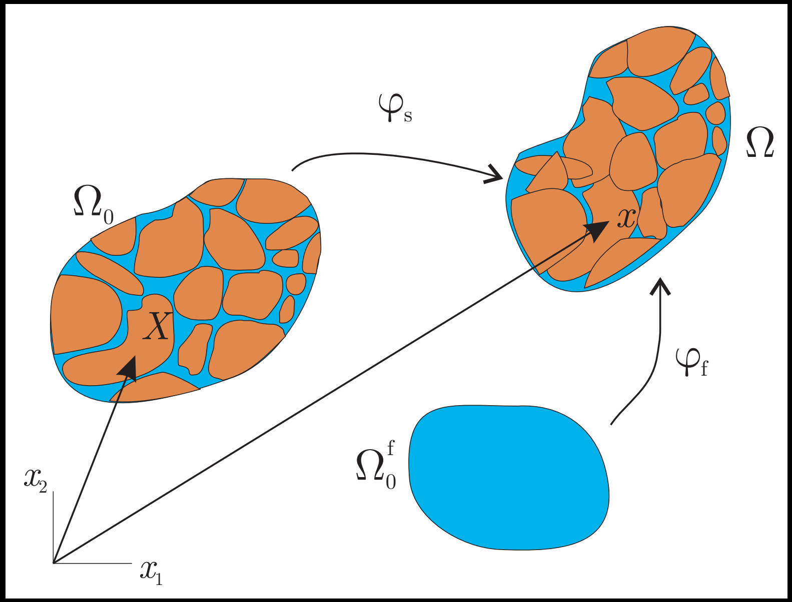
Acceleration output



Soil-fluid interaction

Model ingredients

- Nonlinear continuum mechanics
- Robust constitutive theory
- Computational inelasticity
- Nonlinear finite elements



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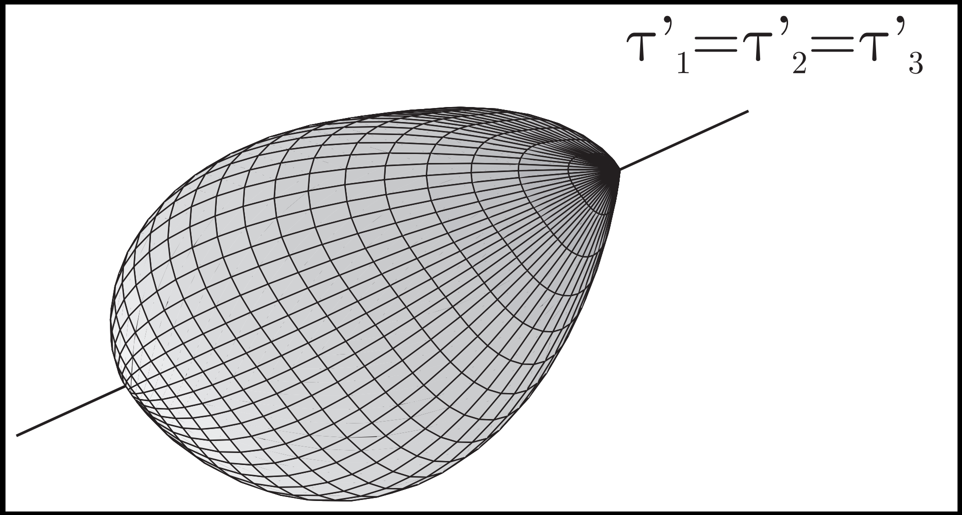

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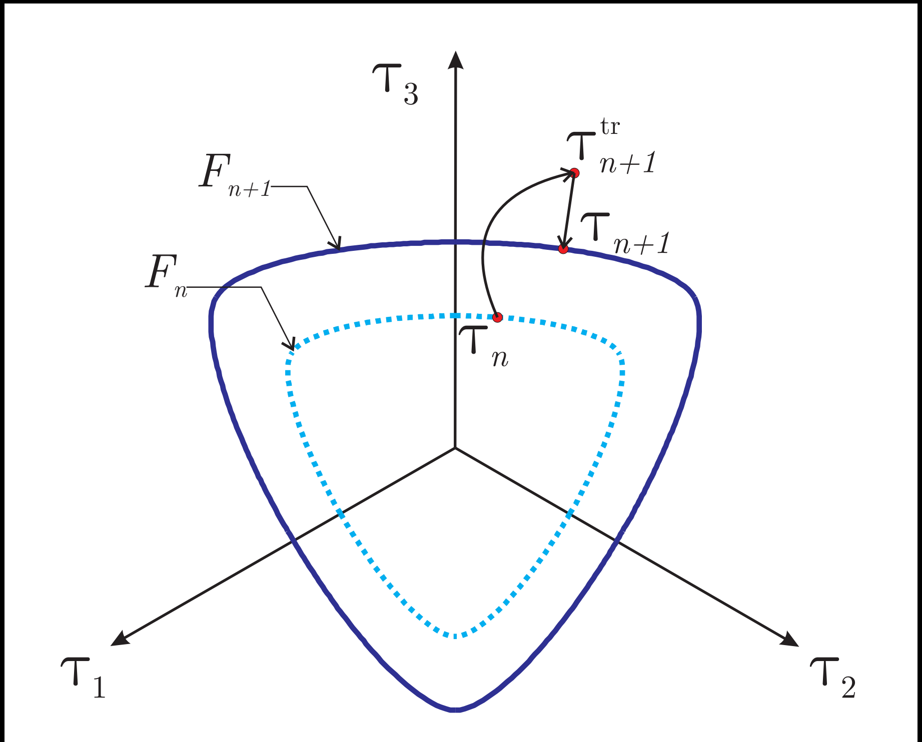

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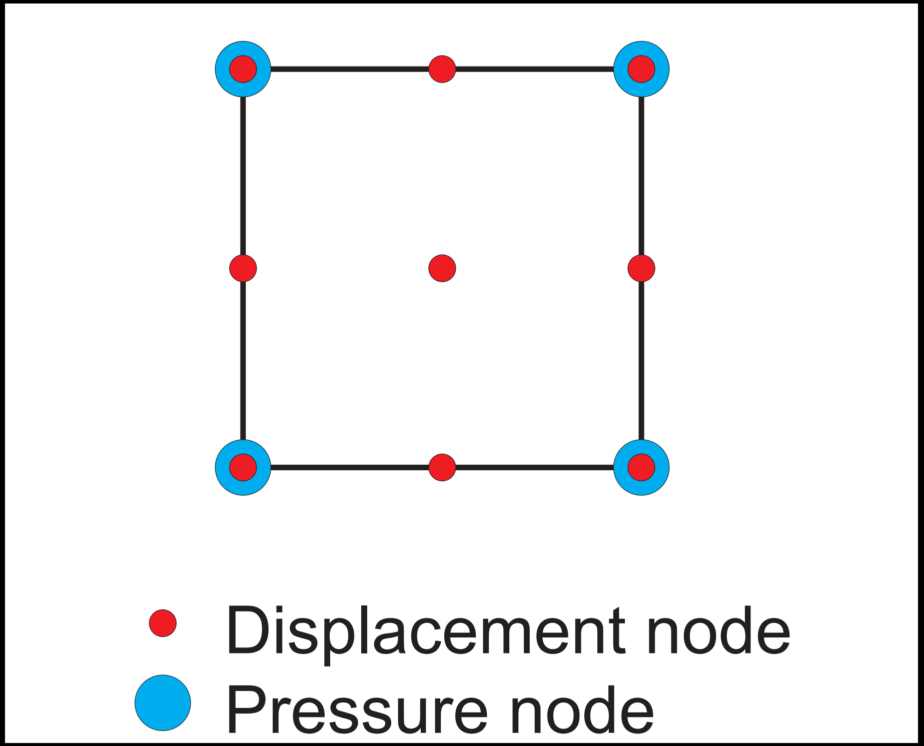
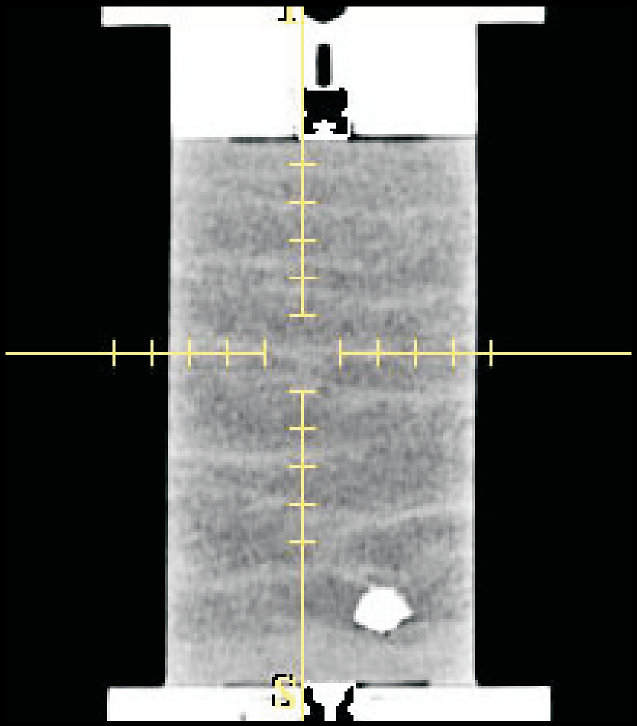
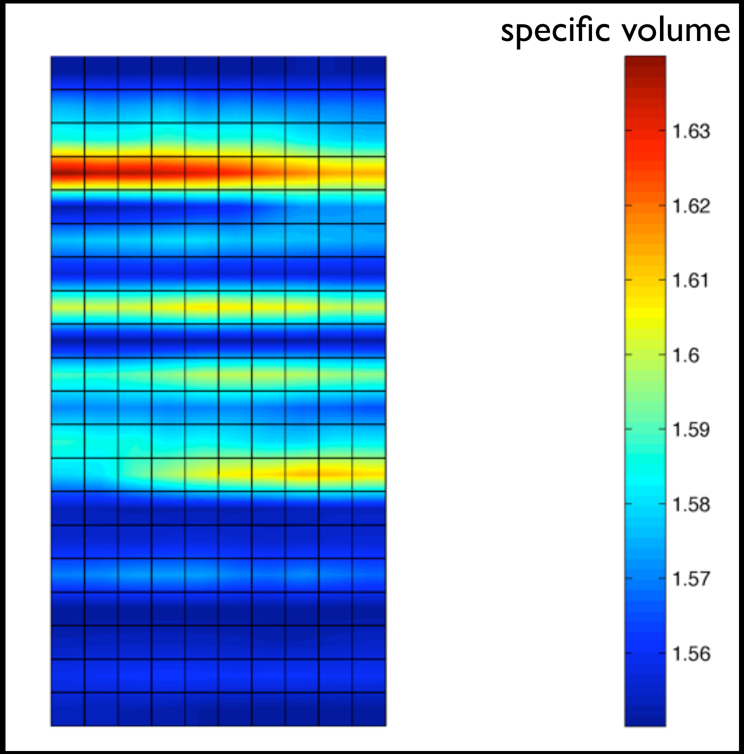


Plane-strain compress

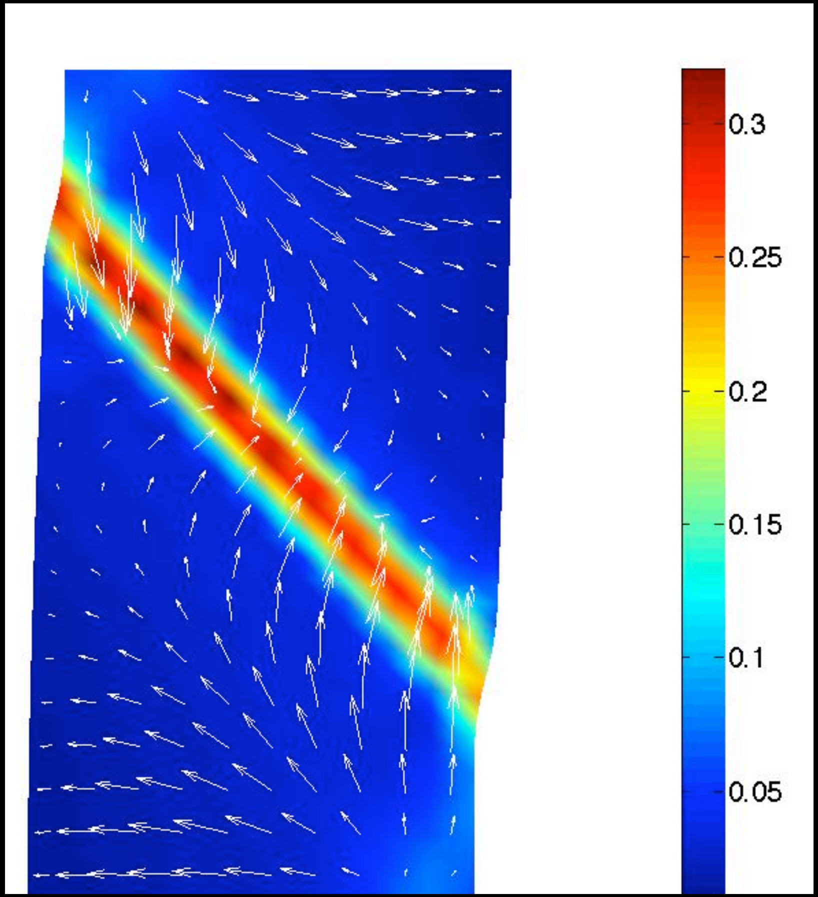


CT scan

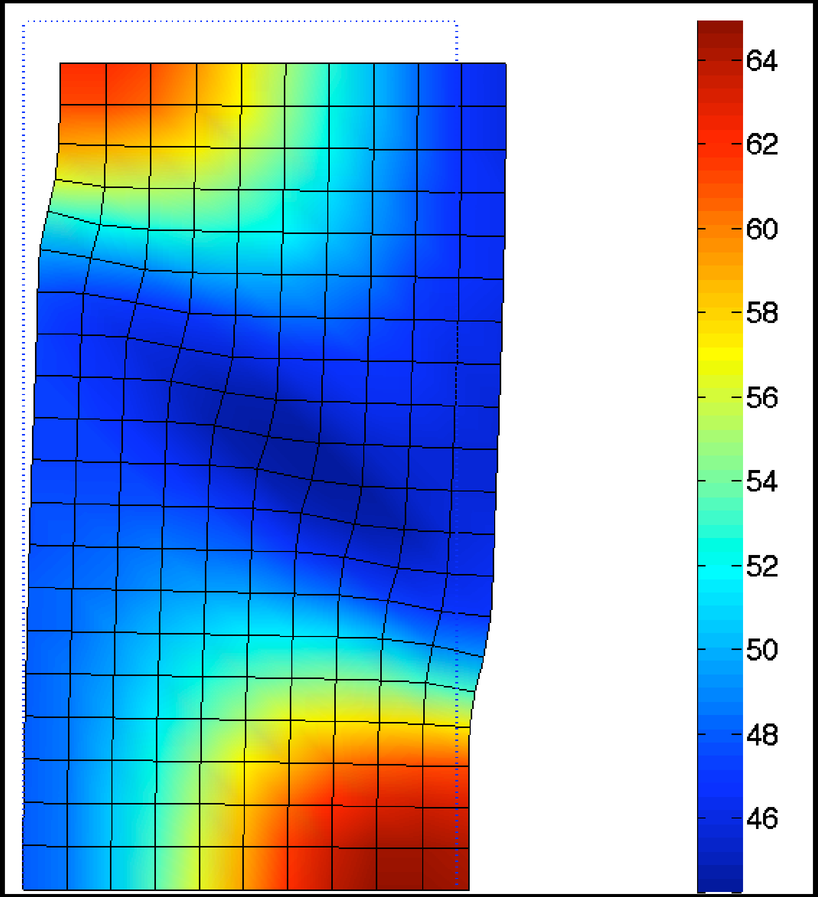


FE model

Plane-strain compress

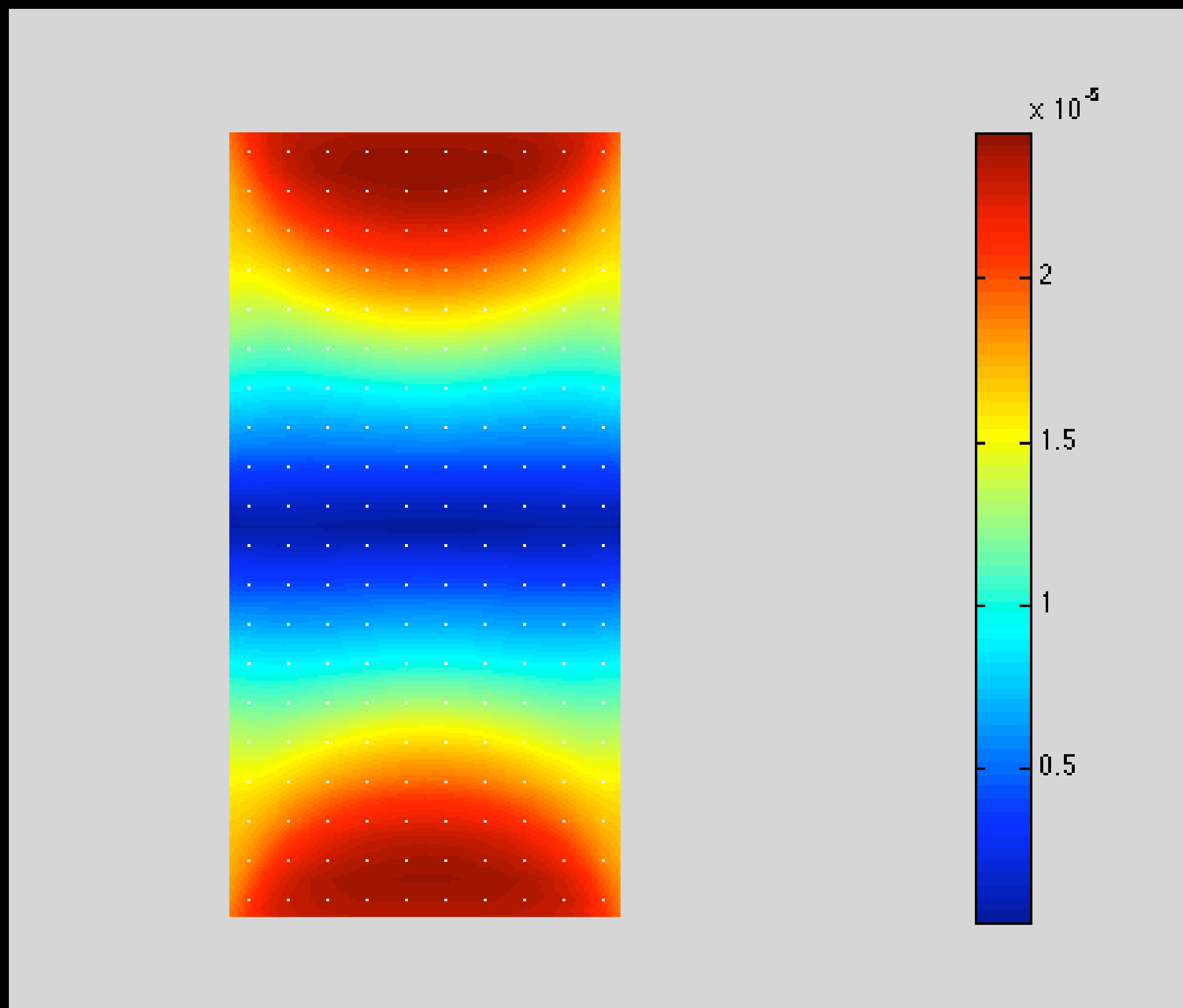


shear strain and flow



fluid pressure

Plane-strain compress



Questions?