Numerical implementation of a novel elastoplastic model for sands based on critical state plasticity

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Motivation

- Critical state plasticity models (e.g. Cam-Clay) provide simple yet effective framework for modeling soil behavior.
- Drawback: Classical Cam-Clay poorly represents sands on ‘dry’ side.
- More accurate representation of soil behavior can be achieved using meso-scale.
- Objective: To develop a simple model based on critical state plasticity accounting for possible inhomogeneities at meso-scale.
- Application: Modeling instability of dense \textit{and} loose sands (e.g. shear band localization).
Hyperelastic model

- Hyperelastic model derived from stored energy function
  \[ \Psi^e (\epsilon^e) = \hat{\Psi}^e (\epsilon^e_v, \epsilon^e_s) \] such that
  \[ \sigma (\epsilon^e) = \frac{\partial \Psi^e}{\partial \epsilon^e} \]

where \( \epsilon^e_v = \text{tr} \ \epsilon^e \) and \( \epsilon^e_s = \sqrt{\frac{2}{3}} \| \text{dev} \ \epsilon^e \| \)

- Resulting invariants of the Cauchy stress tensor
  \[ p = \frac{1}{3} \text{tr} \ \sigma = p (\epsilon^e_v, \epsilon^e_s) \] and \[ q = \sqrt{\frac{3}{2}} \| \text{dev} \ \sigma \| = q (\epsilon^e_v, \epsilon^e_s) \]

- Material parameters: compressibility index \( \kappa \), reference pressure \( p_0 \) at reference elastic strain \( \epsilon^e_{v0} \), initial elastic shear modulus \( \mu_0 \), and coupling constant \( \tilde{\alpha} \)
Plasticity model

The yield surface has the form

\[ F(\sigma, p_i) = q + \eta p \leq 0 \]

where

\[ \eta = \begin{cases} 
M \left[1 + \ln \left(\frac{p_i}{p}\right)\right] & \text{if } N = 0 \\
M/N \left[1 - (1 - N) \left(\frac{p}{p_i}\right)^{N/(1-N)}\right] & \text{if } N > 0 
\end{cases} \]
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Similarly, the plastic potential

\[ Q(\sigma, \bar{p}_i) = q + \bar{\eta} p \]
Plasticity model

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If \( \bar{N} = N \) and \( \bar{p}_i = p_i \) the model is said to be associative.
Typical yield surfaces: meridian plane
Effect of nonassociative flow rule
Flow rule and hardening law

The nonassociative flow rule is defined as

\[ \dot{\varepsilon}^p = \dot{\lambda} \frac{\partial Q}{\partial \sigma} = \dot{\lambda} \left[ \frac{1}{3} \beta \frac{\partial F}{\partial p} \mathbf{1} + \sqrt{\frac{3}{2}} \frac{\partial F}{\partial q} \frac{\text{dev} \sigma}{\| \text{dev} \sigma \|} \right] \]
Flow rule and hardening law

- The nonassociative flow rule is defined as

\[
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\]

- The hardening law is given by

\[
\dot{p}_i = h (p_i^* - p_i) \dot{\varepsilon}^p = h (p_i^* - p_i) \dot{\lambda}
\]

where

\[
\frac{p_i^*}{p} = \begin{cases} 
\exp \left( \frac{\bar{\alpha} \psi_i}{M} \right) & \text{if } \bar{N} = N = 0 \\
(1 - \bar{\alpha} \psi_i N / M)^{(N-1)/N} & \text{if } 0 \leq \bar{N} \leq N \neq 0
\end{cases}
\]

with \(\bar{\alpha} \beta = \alpha\) and \(\beta = (1 - N) / (1 - \bar{N})\)
So, what is $\psi_i$?

State parameter provides info on relative density of soil: meso-scale approach using CT-scan technology.
Maximum dilatancy: $\frac{p_i^*}{p}$

![Graph showing the relationship between $\psi_i$ and $\frac{p_i^*}{p}$ for different values of $N$. The graph displays three curves: $N=0$, $N=0.25$, and $N=0.5$. The curves show a decrease in $\frac{p_i^*}{p}$ as $\psi_i$ increases.]
Numerical implementation

- Fully implicit stress-point integration in strain space i.e.,

\[ r(x) = \begin{cases} 
\epsilon_v^e - \epsilon_v^{e\text{tr}} + \Delta \lambda \beta \partial_p F \\
\epsilon_s^e - \epsilon_s^{e\text{tr}} + \Delta \lambda \partial_q F \\
F 
\end{cases} \]

where \( x = \{\epsilon_v^e, \epsilon_s^e, \Delta \lambda\}^t \)
Numerical implementation

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F
\end{cases}
\]

where \( \mathbf{x} = \{\varepsilon_v^e, \varepsilon_s^e, \Delta\lambda\}^t \)

- Plus, one sub-local Newton iteration to solve for \( p_i \) as it is embedded in the evolution equations. We have sub-local residual

\[
r(p_i) = p_i - p_{i,n} - \Delta\lambda h (p_i^* - p_i)
\]
Algorithmic tangents

Local algorithmic tangent

\[ r'(x) = \frac{\partial r}{\partial x} \]

with trial elastic strains \( \epsilon^e_{v\text{tr}} \) and \( \epsilon^e_{s\text{tr}} \) fixed
Algorithmic tangents

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with trial elastic strains \( \varepsilon_{v}^{e_{\text{tr}}} \) and \( \varepsilon_{s}^{e_{\text{tr}}} \) fixed

- Similarly, the global consistent tangent at time \( t_{n+1} \) is obtained by

\[ c = \left. \frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} \right|_{x} \equiv \left. \frac{\partial \sigma_{n+1}}{\partial \varepsilon_{e_{\text{tr}} n+1}} \right|_{x} \]

where,

\[ \sigma_{n+1} = p_{n+1} 1 + \sqrt{\frac{2}{3}} q_{n+1} \hat{n}_{n+1} \]
Consistent tangent operator

We calculate $c$ using the converged local tangent $r'$. By the chain rule

$$
c = 1 \otimes \left( D_{11} \frac{\partial \epsilon_e^e}{\partial \epsilon} + D_{12} \frac{\partial \epsilon_s^e}{\partial \epsilon} \right) + \sqrt{\frac{2}{3}} \mathbf{n} \otimes \left( D_{21} \frac{\partial \epsilon_e^e}{\partial \epsilon} + D_{22} \frac{\partial \epsilon_s^e}{\partial \epsilon} \right) + \frac{2q}{3\epsilon_s^{e\text{tr}}} \left( \mathbf{I} - \frac{1}{3} 1 \otimes 1 - \mathbf{n} \otimes \mathbf{n} \right)
$$

where, recalling $r$, we have

$$
\frac{\partial x}{\partial \epsilon} = - \begin{pmatrix} \frac{\partial r}{\partial x} \bigg|_{\epsilon_e^{\text{tr}}, \epsilon_v^{\text{tr}}} \\ \frac{\partial r}{\partial x} \bigg|_{r'} \end{pmatrix}^{-1} \cdot \frac{\partial r}{\partial \epsilon} \bigg|_x
$$
\( r' \) and \( c \equiv \) quadratic convergence

![Graph showing iteration number vs. relative residual norm for different step numbers and tolerance levels.](image-url)
$r'$ and $c \equiv$ quadratic convergence
Finally... A BVP: $\nu_0$
Finally... A BVP: $\det A^\text{ep}$
Finally... A BVP: Some comparisons

![Graph showing comparisons between homogeneous and inhomogeneous materials.](image-url)
Finally... A BVP: Some comparisons
Closure

- We have presented a novel nonassociative elastoplastic constitutive model for sands.
- Fully implicit numerical integration using return mapping provides C.T.O. in closed-form.
- Model capable of capturing typical dense sand behavior such as initial compaction followed by dilation.
- Incorporation of meso-scale information (i.e., relative density) allows more accurate representation.
- Model is capable of capturing strain localization of dense sands.
Ongoing/future work

- Three-invariant enhancement on $F$ and $Q$: important to distinguish pure tension/pure compression
- Multi-phase formulation via mixture theory
- Use items above to model shear localization of dense saturated sands
- Recall model capable of modeling loose sands also. Modeling loose saturated sands: modeling liquefaction?