The Tank: Water flow through porous media

Jeff Gellis and Andrew Jacobs

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1 What Exactly, Is Going On?

Despite what some may think, Calculus can be useful outside of mathematics.

Imagine a liquid waste containment pond (sewage pond) with a deadly strain of bacteria. It is summertime and much of this sewage water seeps into the soil with the help of summer monsoon rainfalls. Where exactly will this contaminated water go?

Now, imagine that this sewage pond is adjacent to a large mountain and the that a lake used by a city for drinking water lies on the other side of this mountain. Usually the water level of the lake is higher than that of the sewage pond, but monsoon rains have filled the sewage pond over capacity; it is now higher in elevation than the lake and its contaminated waters are starting to seep into the soil at an accelerated rate (see Figure 1). This is happening in an arid region with very sandy soil, so drinking water is a very important issue in this area. A similar, real-life situation is shown in Figure 2.

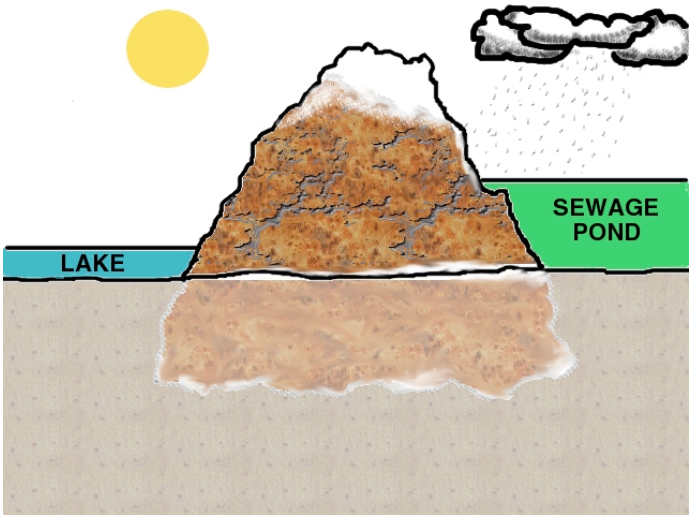

Figure 1: A diagram of our scenario

Some of the more intelligent townspeople (engineers) are worried that the water from the sewage pond could flow into the lake. They propose either diverting the lake into a secured reservoir or

Figure 2: A real-life example of this situation.

draining the sewage pond into a dry lake bed next to the pond; they believe either project would mitigate the problem. The local government (made up of political-science majors), however, does not believe that the sewage pond will drain into the lake because a large mountain separates the two bodies of water; they find it difficult to believe that water can flow around a mountain. The government also does not want to invest in a large scale project involving the lake because it would be expensive.

Which group is right? Can we be sure that the sewage water will flow into the lake? How can we balance science, ethics, and cost all at the same time? Could calculus help solve the problem?

You have probably figured out that the contaminated water has the potential to make its way to the lake where it could wreak havoc on the town. How could this tragedy be averted? Thankfully, calculus does in fact become useful. It is at our disposal for us to model and predict situations like this.

Now imagine that you are the engineers living in this town. You have devised a model that you think will show you the path that the sewage water will take. From there you hope that the results will prompt action on the national level, if needed, to intervene in the situation.

2 How Do the Concepts Apply?

It may seem unlikely that concepts from calculus can remediate a virulent strain of bacteria, but in fact, they have the power to indirectly solve the problem.

The first concept that must be understood is that the contaminated water will flow from a higher potential to a lower potential. In this case, it means that the water will flow from the water in the sewage pond (which has a higher water level) to the water in the lake (which has a lower water level). Water will continue to flow from high to low until equilibrium is achieved. At equilibrium, the water levels of both the sewage pond and the lake will be at an equal elevation. When this happens, the flow between the two bodies of water will stop. You most likely have seen variations

of this concept in other disciplines such as physics, thermodynamics, and chemistry (heat diffusion, molecular entropy, LeChatelier's principle, etc).

The second concept to understand is that the water will interact with the soil (sandy soil in this case) by diffusing through things called "flow channels." These channels are the tiny voids in between particles of soil (see Figure 3) used by fluids to move through porous media.

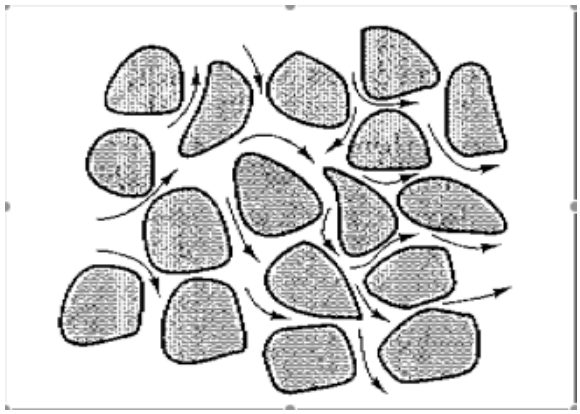

Figure 3: Fluid flow through soil particles

The cool thing is that we can predict what the flow of the contaminated water will look like underground (pretty hard to do with your eyes unless you are Superman) and even approximate its trajectory and path around underground obstacles such as subterranean cliffs. We can approximate these properties with something called the **Laplace Equation**:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{1}$$

In the Laplace equation, Φ stands for a scalar function. Since the equation is useful in many different fields, several functions could be used in place of Φ. The function h for total head (used in fluid dynamics) and T for temperature (used in thermodynamics) are just two examples.

Another point of interest in the Laplace equation is the ∂ symbol. This notation is used to refer to a partial derivative. When you have more than one independent variable in calculus, sometimes you only want to take a derivative with respect to one variable. This is called a partial derivative. For example, if you had the equation: $f(x, y) = x^2y + 2y$ and you were asked to evaluate $\partial f/\partial y$, you would be taking the partial derivative of the function f with respect to y. To do this, you treat y like a variable and x like a constant while differentiating. The result would be $\partial f/\partial y = x^2 + 2$. We can also take the partial derivative of f with respect to x. That result would be $\partial f/\partial x = 2xy + 0$.

Since we are studying fluid dynamics, we will use this version of the Laplace equation:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{2}$$

In the equation, x represents horizontal distance, y represents vertical distance, and h represents total head, or the potential at a point. In this case, it is the height of the water-bodies above ground level. Since the partials are squared, it means that we are taking the *second* partial derivatives of total head with respect to both x and y.

As mentioned above, the Laplace equation can be used to model a lot of different things (e.g. fluid flow, heat flow, and electrostatics). However, we will use it mainly to model the flow of contaminated water from a higher elevation to a lower elevation (the sewage pond scenario). Later in this handout, we will also use the equation to model simple heat flow.

3 How Do We Model This?

Although our real-world scenario is plausible, it would be hard to reproduce in a laboratory setting. Thus, we will use a tank to simulate this potential difference (see Figure 4). The tank was constructed such that an elevation difference between the two sides of the tank could be established. You can think of the left side of the tank as the lake basin and the right side of the tank as the sewage pond. A divider (a pseudo-mountain) was placed into the tank to simulate a subterranean rock formation (this would be created from the underside of the mountain). The soil is being simulated by a course sand and the levels of sand are about the same in each side of the tank.

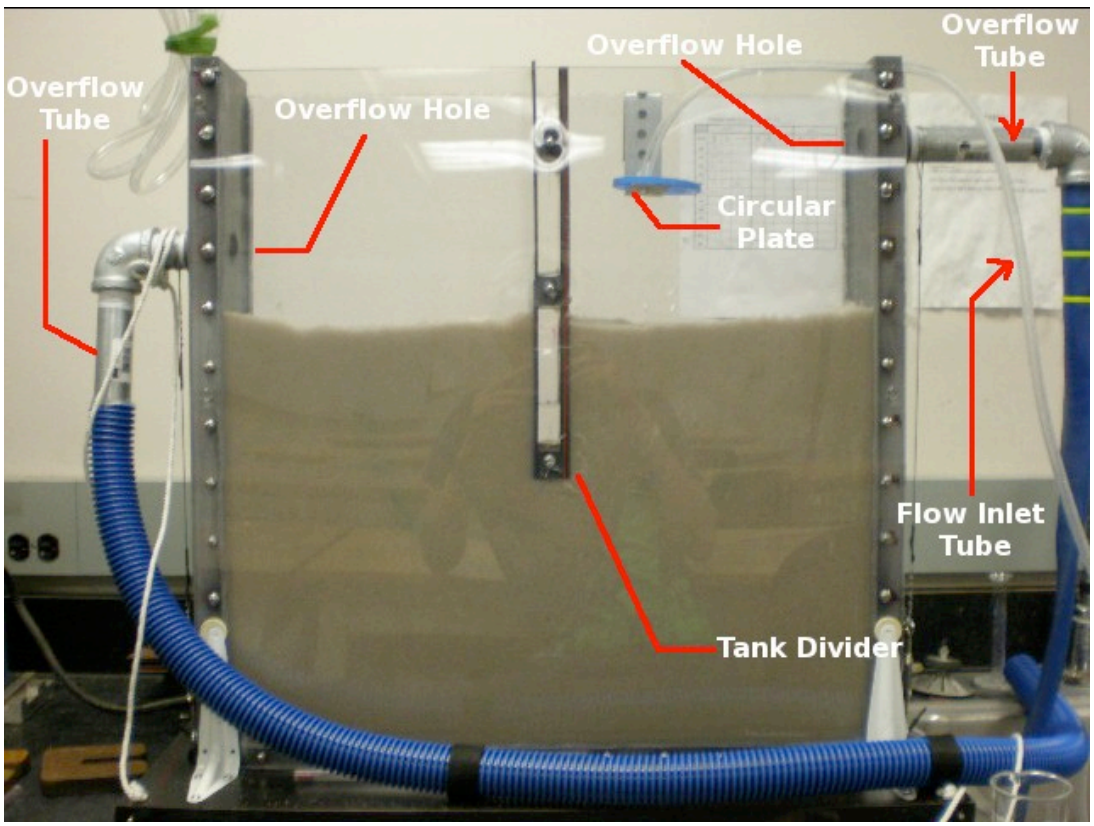

Figure 4: Aparatus to be used in our experiment

To create a visual representation of the flow through the tank, we will use 4 types of food coloring to model the path that the water travels. The results will be very perceivable as food

coloring vibrantly models the flow.

If you remember in the previous section, the potential difference between the two waterbodies is the driving force behind the flow and is perhaps the most crucial concept to understand. To maintain this potential difference, we will be having a constant flow of water into the right side of the tank which, when filled completely, will be at a higher potential than the left side of the tank (see Figure 5). This ensures that the water will flow from the right side of the tank to the left side of the tank.

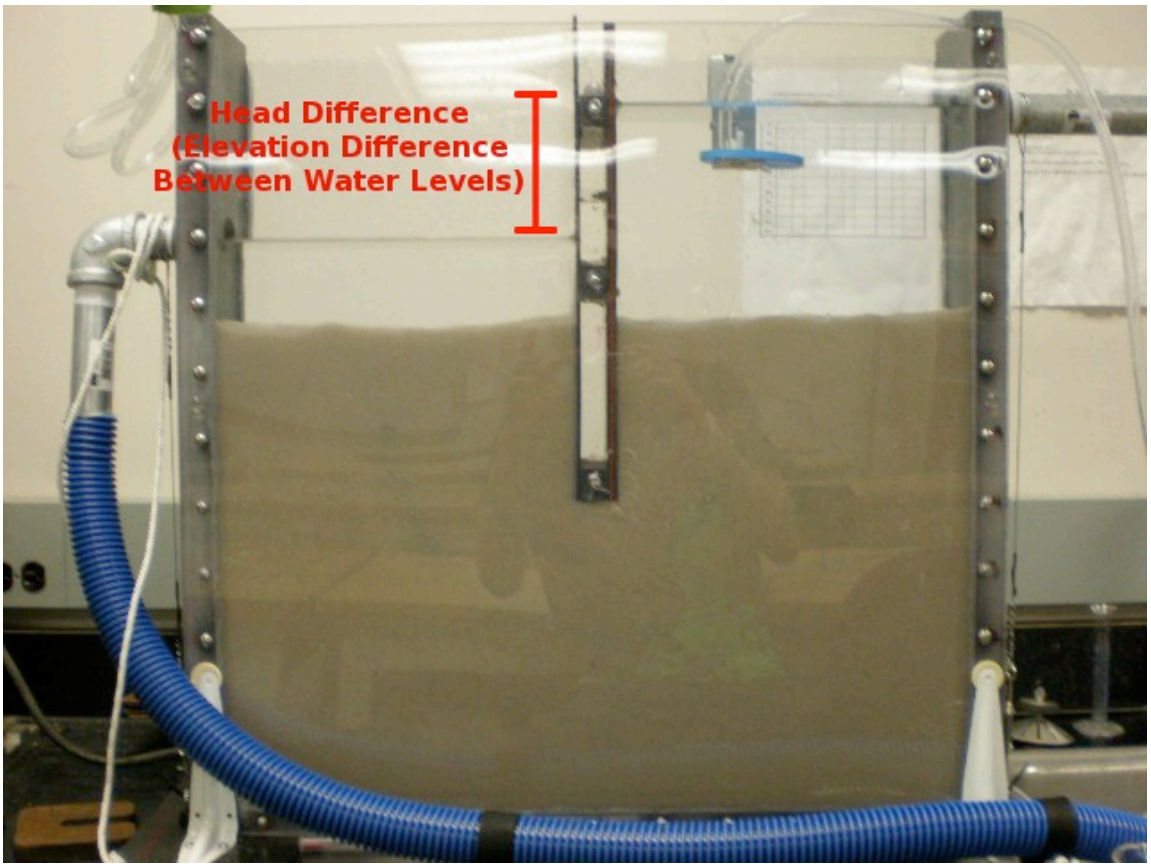

Figure 5: The potential difference between the two bodies of water drives the flow

Major Assumptions:

1. Steady-state: This means that there is no time dependence involved with the flow. This property is inherent in the Laplace equation because there is no time term involved.

2. Two-dimensional, ideal flow: We assume that flow is only in two dimensions (x and y) and that the flow is undisturbed.

3. Potential difference remains constant: The constant flow of water through the tank ensures this potential difference and makes sure that water flows from right to left at a constant rate.

4. The liquid that is flowing is inviscid: An inviscid liquid flows smoothly. A viscous material would be something like molasses. Water is known to be an inviscid, so this is a valid assumption.

5. The sand is adequately saturated: The sand must be saturated with water and have no air pockets to ensure that the flow is undeterred and predictable.

4 How Does It Work?

Now we are ready to simulate this phenomenon with the tank model.

Procedure

1. Make sure sand is saturated with water prior to starting experiment. This is performed simply by filling the tank until the water levels ascend higher than the sand levels, and any excess air is dislodged from the sand manually. The air can be dislodged by disturbing the sand using a long tool or metal rod.

2. If the overflow tubes of the tank are not set to drain into the sink already, make sure to put them in the sink so that they do not drain on the table.

3. Fill both sides of the tank with water by using the flexible pipe connected to the faucet, make sure you use the blue circular plate on the ledge of the tank (see tank diagram) to diffuse the flow of water. This prevents the sand from kicking up.

4. Make sure both sides of the tank are filled up to the maximum level. Once water begins to exit through both overflow holes, this step is complete.

5. Locate the 4 thin plastic tubes by the tank as well as four different colors of food coloring.

6. There should be magnets on both the tank and the tubes. Affix the tubes to the tank at each position where there is a magnet and make sure that the end of the tube with the piece of sponge in it is inserted about a quarter of an inch into the sand. The tube should be inserted as close to the plexiglas siding as possible.

7. Fill each of the tubes nearly to the top with a different type of food coloring, this food coloring will act as our tracer in determining the flow through the tank.

8. If the sink is not turned on and flowing water into the tank, turn on the sink so that there will be flow. Continue to use the circular plate to decrease the flow's turbulence.

9. After about two minutes you should notice the food coloring seeping into the sand. To maintain the flow of the food coloring, you must keep the level of the food coloring in the tubes above the water level in the right side of the tank (another application of potential difference and its effect on flow).

10. About five minutes after putting the food coloring in, the flow lines should become apparent. Wait until they exit the sand on the left side of the tank before taking measurements.

11. Measure the lowest point of each flow line from the bottom of the tank, the distance of the start and end of each flow line from the side of the tank (the flow lines should start and stop at roughly symmetrical points). Also, measure the height of the sand during the experiment and the distance between the water level and the sand on each side of the tank. Then, take digital photos of the tank from straight on. The dimensions of the tank are given below.

Spacial Property	Measuremant (cm)
Length of Tank (x-direction)	53.8
Height of Tank (y-direction)	58.9
Thickness of Tank (z-direction)	10.5
Length of Tank Divider (x-direction)	3.0
Height of Tank Divider (y-direction)	34.8
Length of Left Chamber of Tank (x-direction)	27.1
Length of Right Chamber of Tank (x-direction)	26.7

5 What Are The Results?

When you have completed the experience, your results should look similar to those in Figure 6.

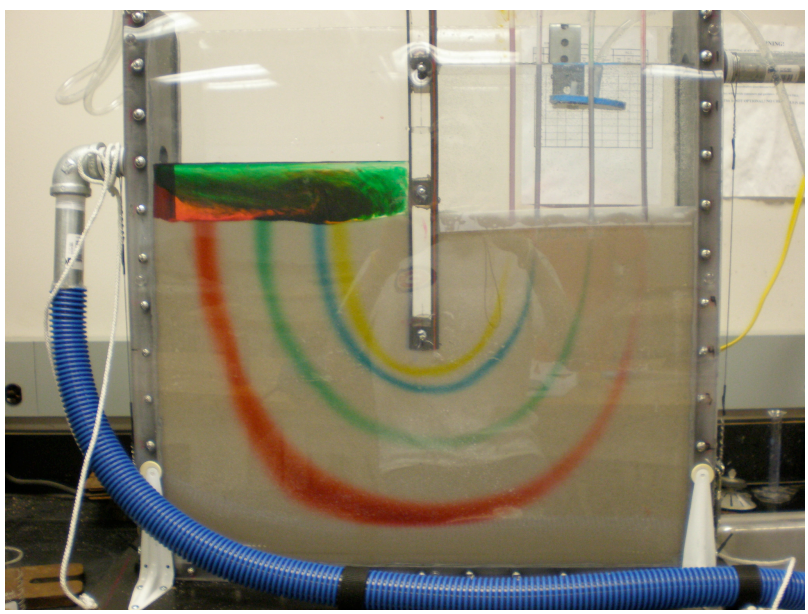

Figure 6: Food coloring models the flow of the water through the tank

The flow path seems to be following a parabolic trajectory, but keep in mind that water flow through soils does not always look like this. The fact that the sand levels are at the same height and that a linear boundary divides the tank at the middle have created the symmetry necessary to produce these parabolic trajectories. In real life, groundwater flow would be much more complicated because of non-uniform soils, underground caverns or tunnels, and other impedances.

6 How Does The Calculus Come Into Play?

So, imagine that we did not have any food coloring. How could we know the direction that the water will flow? In our pond/lake scenario, it would not make sense to have food coloring put into giant tubes and inserted into the bottom of the sewage pond to track the flow. We must use a more practical method. So, we turn back to the Laplace equation:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

The equation becomes a powerful tool because it only really requires a couple of measurements in order to model the flow. In this case, all we need are the geometric properties of the system (tank dimensions, water levels, etc). The solution is not affected by the material properties of the fluid or the medium.

Now you might be asking yourself, "How can we use this equation if we do not have a function to take derivatives of with respect to space?" To do this, we use a process called the "finite difference method." This method is effective in mimicking a function by using approximations of its derivatives. In our case, the finite difference method is used to solve a second-order partial differential equation (the Laplace equation).

An easy way to be introduced to finite differences is through solving the relatively simple case of heat flow in a plate. The equation that governs heat flow at steady state is again, the Laplace equation. Most of the time when we are dealing with differential equations, we will have boundary conditions to keep in mind. For example, in a heated plate the boundary conditions would be the temperatures at the edges of the plate. Once you have both the boundary conditions and the Laplace equation, you can solve for what the flow of heat will look like in the plate.

The first step in using finite differences is to set up something called a mesh. A mesh is simply a piece of geometry that you use to break up your physical conditions into smaller pieces. You can use any shape you want to create a mesh, however, squares are the easiest to use. When constructing a mesh the end result will be a grid. Each point that is on the mesh grid is called a node. The nodes are the fundamental basis behind finite differences. We want to approximate the derivatives at each nodes so that we can use the derivative at that point to approximate the value of the next node. In the case of the heated plate, we are modeling a function of temperature across the mesh. When finite differences are applied to the tank model, we are approximating a function of the total head through the tank.

7 Heated Plate Finite Difference Exercise

We will first make our mesh to discretize (divide up) the domain of the plate. The physical plate is an $L \times L$ square of varying temperature (see Figure 7). The plate can be broken up into an $N \times N$ square mesh. As we increase the value of N, our model becomes more accurate. A generic node created in the mesh will be designated by the value (i, j) (see Figure 8). The temperature at the point (x_i, y_j) is represented by $T(x_i, y_j)$ or $T_{i,j}$.

The physical boundaries show that there is a temperature of 0 along the upper, lower, and right boundaries of the plate and a temperature of 1 along the left boundary. The heat from the left side dissipates through the plate, creating the flow. Once the system reaches a steady state equilibrium, it can be modeled by the Laplace equation. As mentioned before, the flow can be modeled using

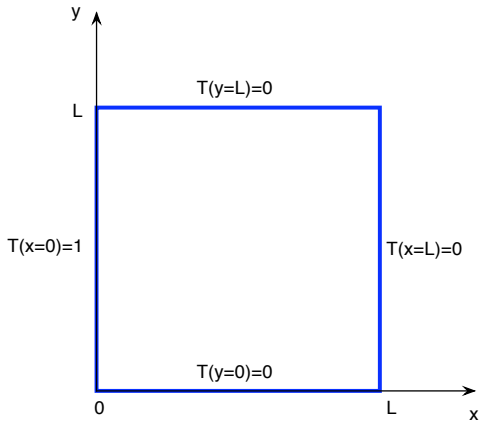

Figure 7: The metal plate of size $L \times L$

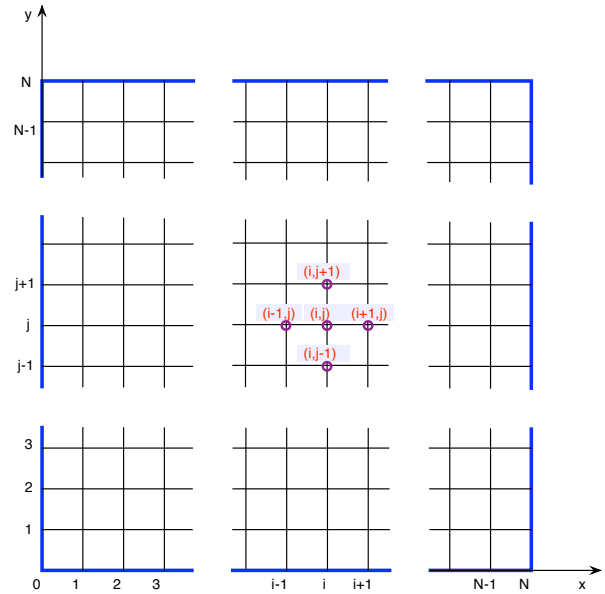

Figure 8: The plate transformed into an NxN mesh

the Laplace equation and the boundary conditions. This process begins using the approximation of a partial derivative:

$$\frac{\partial T(x_i)}{\partial x} \approx \frac{T(x_i) - T(x_i - 1)}{\Delta x} \qquad ; \qquad \Delta x = x_i - x_{i-1} \tag{3}$$

It follows that

$$\frac{\partial^2 T(x_i)}{\partial x^2} \approx \frac{\partial\left(\frac{T(x_i)-T(x_i-1)}{\Delta x}\right)}{\partial x} \approx \frac{\Delta T(x_i) - \Delta T(x_i - 1)}{\Delta x^2} = \frac{(T(x_i) - T(x_{i-1})) - (T(x_{i-1}) - T(x_{i-2}))}{\Delta x^2}$$

We can manipulate the Laplace equation using this definition to create simpler conditions. So,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \approx \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta x^2} + \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{\Delta y^2} \tag{4}$$

Since our mesh is a square mesh, $\Delta x = \Delta y$ and the sum of the two numerators is equal to zero. Once this condition is added to our boundary conditions, we get:

$$T_{i-1,j} - 4T_{i,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1} = 0 \tag{5}$$

$$T_{0,j} = 1 \tag{6}$$

$$T_{N,j} = T_{i,0} = T_{i.N} = 0 \tag{7}$$

When $N = 4$ is plugged into these equations, the following matrix equation can be created.

$$
\underbrace{\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix}}_{\text{Coefficient matrix}} \underbrace{\begin{Bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{1,2} \\ T_{2,2} \\ T_{3,2} \\ T_{1,3} \\ T_{2,3} \\ T_{3,3} \end{Bmatrix}}_{T\text{ vector}} = \underbrace{\begin{Bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{Bmatrix}}_{\text{Solution vector}} \tag{8}
$$

Each row in the coefficient matrix represents the condition around the node given by the respective entry in the T vector. For example, the third row of numbers represents the equation $T_{2,1} - 4T_{3,1} + T_{3,2} = 0$. These are the conditions around the third node in the T vector, $T_{3,1}$.

This is the matrix equation you will obtain if $N = 4$ is your mesh size. As you can probably tell, there is a pattern in this equation, specifically in the coefficient matrix and solution vector. The equation changes based on the value used for N. The dimensions and patterns of the coefficient matrix also depend on N. One example of this dependence is that the dimensions of the coefficient matrix are $(N-1)^2 \times (N-1)^2$ for any selected value of N. There are several patterns within the coefficient matrix and solution vector and it is left as an exercise for you to find them. These patterns hint at the fact that a general matrix equation can be constructed for any value of N greater thatn 2.

What we are interested in mostly is the T vector. This is because the T vector gives us the temperature at each node as seen above. Using basic matrix algebra, you can obtain the T vector. When this T vector is modified into a matrix and plotted in the correct interval, you can obtain a contour plot of heat flow across the plate (see Figures 9 and 10).

To modify the T vector so it plots correctly, you must assemble it into a $(N-1) \times (N-1)$ matrix. The T vector is split up into a square matrix. Then, zeroes are placed along the top, right, and bottom sides. Ones are added only to the left side but not on the corners. The zeros and ones are added to represent the temperatures at the boundaries. Here is the example for $N = 4$.

$$
\begin{bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{1,2} \\ T_{2,2} \\ T_{3,2} \\ T_{1,3} \\ T_{2,3} \\ T_{3,3} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & T_{1,1} & T_{2,1} & T_{3,1} & 0 \\ 1 & T_{1,2} & T_{2,2} & T_{3,2} & 0 \\ 1 & T_{1,3} & T_{2,3} & T_{3,3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}
$$

Once you have modified the T vector into an $(N-1) \times (N-1)$ matrix, you can plot it versus an x-vector and a y-vector. Both vectors are composed of values evenly spaced between 0 and L and have the same number of values as the T matrix has columns. Figrues 9 and 10 contain examples of graphical solutions with different mesh sizes.

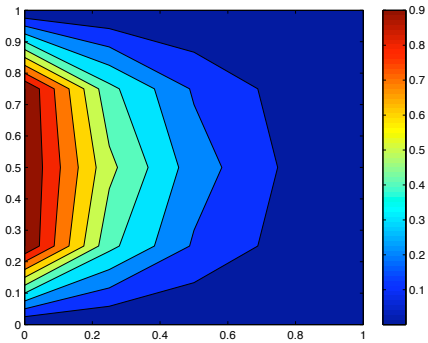

Figure 9: Contour plot of T for $N = 4$

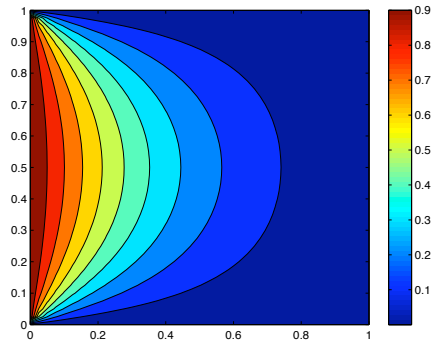

Figure 10: Contour plot of T for $N = 40$

You may be wondering what this heat contour plot means physically. The plot is a $2D$ representation of the $3D$ graph of x versus y versus T. The contour lines are a collection of points that have the same temperature. As the colors become cooler, so does the temperature of the plate at that point.

Now you might ask, how do we solve for all these matrices and get graphical results? The answer to that question is a piece of software called Matlab. If you are not familiar with Matlab, it is a powerful computer program principally designed to take advantage of matrices. The program is very useful in solving larger matrix equations, arrays, and other mathematical quandaries.

To get started on the coding for this heated plate experiment, some incomplete code is being provided for you in heatedplatecode.m in this folder. You may not be familiar with the coding structure of Matlab, so only the mathematical concepts of the code will need to be filled in. At this point, you should use this template in Matlab to create your own program that produces a contour plot for *any* value of N. If you wold like to take a shot at coding the solution for yourself, feel free.

8 How Do We Solve For The Flow In The Tank?

We can solve for the flow in the tank numerically by again using the power of finite differences. We can set up a mesh and a matrix equation and use Matlab to code a program that plots the flow for us. However, this process is much more complex due to the "no-flow" nature of the tank boundaries. In the plate problem, the heat was free to flow out through the three cooler boundaries. With the tank, no water can exit through the sides of the tank nor can it pass through the divider. Because this process is so much more complicated, we will solve for the flow in the tank using a Matlab applet called pdetool (aka Partial Differential Equation Toolbox). Pdetool lets us define our boundary conditions visually within the applet by drawing them. This makes solving for the flow within the tank much easier. Please see the Appendix for step-by-step instructions for using pdetool.

The culmination of the pdetool process is a graph of the flow lines and equipotentials for the Tank such as the one in Figure 11. In that picture, the arrows that are traveling from right to left are velocity vectors of the flow. The lines that are perpendicular to these arrows are called the equipotentials. Equipotentials lines that contain points with the same potential. As discussed

earlier, flow occurs because fluids move from higher potential to lower potential. are The next picture, Figure 12, is an overlay of the solution from pdetool onto a digital photo that we took of our tank during the experiment. The results are very similar when you compare the two.

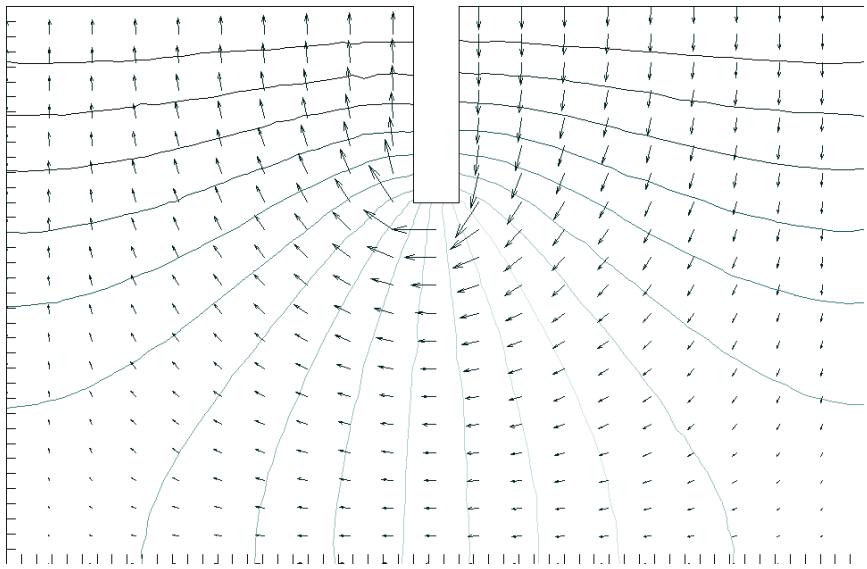

Figure 11: pdetool can give us flowlines and equal potentials for the tank system.

What we have just done is pretty amazing when you think about it. We were able to model the flow of a liquid through the ground, something that is usually very difficult to do, with only two geometric parameters (the head in each tank along with the boundary conditions of the tank). It may not be clear at this point, but the ability to model this has profound real-world applications. Let us return to our scenario from earlier.

9 What is Going to Happen In The Real World?

The tank model is pretty close to the setup of the real world scenario posed earlier. In both cases we are dealing with a medium that is uniform and porous (sand). The two also share similar impedances to flow (the tank divider vs. the large mountain) and potential differences created by the amount of water on each side of the impedance (see Figures 1 and 5).

If we retain some of the basic assumptions with the tank model and apply them to the real world scenario, we can make a homologous comparison between the two. The sewage pond scenario lends itself to this model because there are only a couple assumptions that have to be made and these assumptions are fairly realistic and reasonable.

10 Conclusion

After you and your engineering colleagues run the tank model experiment and solve it analytically, it becomes apparent that the water from the sewage pond *will* drain into the lake. Hence, you

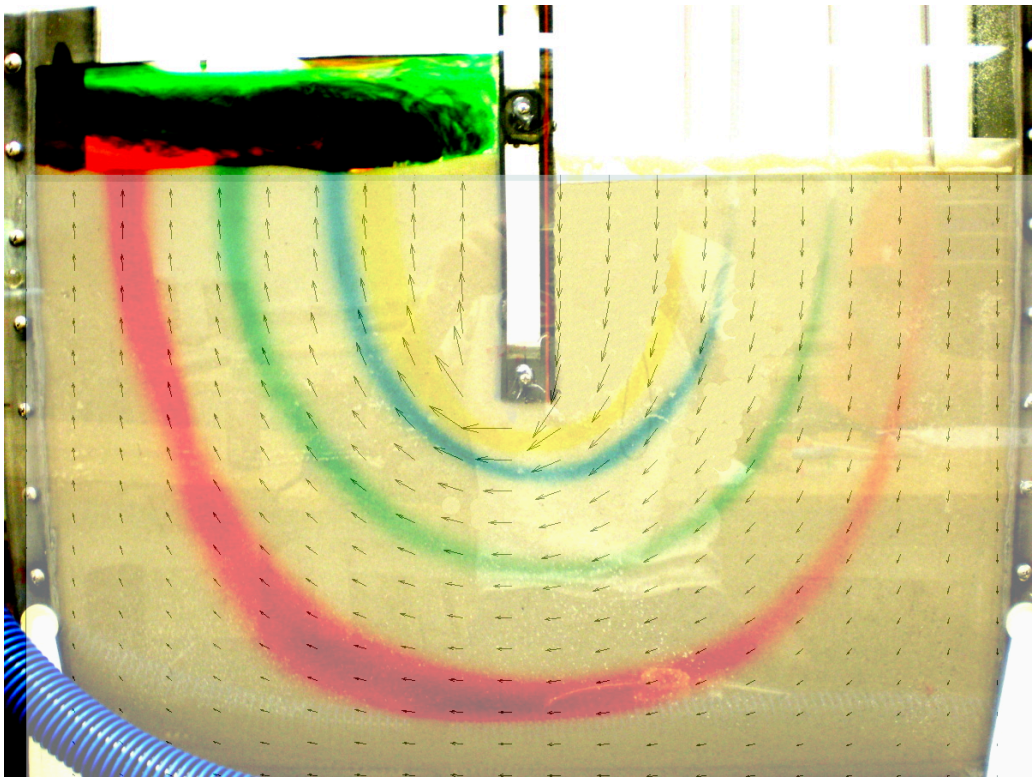

Figure 12: Our experimental results and our numerical results should match up nicely.

contact the state officials and they set a plan in motion for prompt action to avert the disaster.

Unless of course, one of your colleagues (classmates) disagrees....(This is where the narrator is actually talking to you, the audience...do you agree with the conclusions of this experiment? Why or why not?)

Questions for you:

1. If you agree with the results, what course of action would you take to make sure the sewage pond water does not reach the lake used for drinking water? Would you divert the lake into a secured reservoir? Would you drain the sewage pond into a dry lake bed next to the pond? What are the pros and cons of each situation? Any other ideas?

2. What fundamental concept (explained earlier) is at work in determining the direction water will flow through ideal soil when two water bodies are separated? With this concept in mind, why was the sewage pond not an issue before the monsoon rainfall came (refer to story)?

3. Explain why finite differences are a good technique to approximate the water flow through

the soil. Why are we using these numerical approximations to solve for the flow?

4. Do you think that water would still flow from the sewage pond to the lake if the sand level in the sewage pond was lower than the sand level in the lake? Explain.

5. Imagine that the town was able to bioremediate the sewage pond and decided to let it flow into the lake. When would the water from the "clean" sewage pond stop flowing into the lake? What is the generic term of this phenomenon called?

6. Extra Credit: Using the concepts introduced in this experience, can you devise a way to drain the tank (after you are done with the experiment) with only the flexible plastic tube connected to the sink?

11 What Have We Learned?

The point of this experience was to show you how the concepts you learn in school have real-world application and that they can also be modeled in a laboratory setting.

Sometimes you question the validity of the material you learn in the classroom and its usefulness in your future life. Hopefully, seeing how the Laplace equation (something that is seemingly insignificant on paper) can model flow will give you a glimpse of how calculus is relevant in the real world. There are many other equations out there that you may be familiar with, but it takes some thinking to apply them to new and more useful situations.

WARNING EXTREMELY CORNY STATEMENT TO FOLLOW

With that said, go out in the world and discover something new.

Appendix: pdetool instructions

1. Set the axes limits in the options menus. Set the limits so that the dimensions of the *sand* can comfortably fit within the graph. We only need to draw the sand because this is where we mapped the flow using the food coloring.

2. Change the grid spacing and turn snap on also in the options menu. The spacing is the number in the middle of the two colons. Change it to 1 for both x and y.

3. Use the first five buttons to create the physical geometry of the system. The most useful tool for creating the tank is the 5th button, which allows you to create a custom polygon. Again, you only need to draw the sand.

4. Click the 6th button to go into boundary mode. You can double click on each boundary to set the conditions. A no-flow boundary (any side or bottom of the tank or tank divider) is a Neumann boundary with both g and q equal to zero. The two flow boundaries (the top of the sand) are Dirichlet boundaries with h equal to one and r equal to the total head. In our case, the total head is the distance between the top of the water and the top of the sand.

5. The 7th button allows you to set the PDE conditions. The equation given by the program is a generic Laplace equation. Our version has $c = 1$, $a = 0$, and $t = 0$.

6. Once the Laplace equation is specified, click the 8th button once to create a mesh (this program uses triganles, not squares, for its meshes) and the 9th button once or twice (the more you press this refine mesh button, the more accurate your solution will be, but the slower it will run).

7. After this, press the = button to solve the problem. Then press the final button to set the options for the graph.